Infinite games and Lebesgue measure

Bird's Eye Conference 2024

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23 Mar 2024

Infinite games

The simplest of an infinite game is the following: Given some set $X \subseteq [\mathbb{N}]^\infty$, consider the following two-player game:

I	n_0		$n_2 > n_1$		$n_4 > n_3$	$\cdots$
II		$n_1 > n_0$		$n_3 > n_2$		$\cdots$

where $n_0, n_1, \ldots$ are natural numbers. **I** wins iff $\{n_0, n_1, \ldots\} \in X$. Let's call this game G_X. Note that you can't draw in this game.

Natural question:

For any set X, must one of the players have a winning strategy to win the game G_X?

Definition

A set $X \subseteq [\mathbb{N}]^\infty$ *determined* if one player has a winning strategy in G_X.

Determined sets are easy to construct.

Example

$\emptyset$ is determined, as Player **II** has a strategy by literally doing anything.

Example

Let X be the set of all infinite $A \subseteq \mathbb{N}$ such that $0 \in A$. Then **I** has a strategy by playing 0 on the first turn, then take a vacation.

Recall that the Axiom of Choice (AC) asserts that every set can be well-ordered.

Theorem (Gale-Stewart)

AC *implies that there exists some infinite* $X \subseteq [\mathbb{N}]^\infty$ *which is not determined.*

Sketch of Proof.

A strategy can be thought of as a function $\sigma : \mathbb{N}^{<\infty} \to \mathbb{N}$, since it reads what the other player has played so far (which is a finite sequence of naturals), and outputs a natural for the player to play on a specific turn. Using AC, we well-order all possible strategies from both players, and construct X by diagonalising all such strategies. □

Axiom of Determinacy

But what if we forget about AC?

Definition

The *Axiom of Determinacy*, also written as AD, asserts that the game G_X is determined for all $X \subseteq [\mathbb{N}]^\infty$.

Note that this implies that many other types of games are determined, as long as **I** and **II** play objects from a countable set.

While AD is incompatible with AC, it turns out that AD implies a weak form of choice.

Lemma (ZF + AD)

The Axiom of Countable Choice, AC_ω, holds.

The actual statement is not important. However, measure theory works best with AC_ω. For example, AC_ω implies that the union of countably many countable sets is countable.

AD and Measurability

Figuratively, everybody knows that AC implies the existence of a non-Lebesgue measurable set.

Theorem (ZF + AD)

Every subset of $\mathbb{R}$ is Lebesgue measurable.

The outline of the proof is as follows:

(1) It suffices to show that, for every $S \subseteq \mathbb{R}$ such that every measurable subset of S is null, S is also null.

(2) Fix some $S \subseteq [0, 1]$ such that every measurable subset of S is null. It suffices to show that $\mu^*(S) \leq \varepsilon$ for all $\varepsilon > 0$.

(3) We introduce the *covering game*. **I** plays a binary sequence which gives us a real number a, and **II** plays open sets which tries to "cover" a.

(4) We show that **I** can't have a winning strategy. By AD, **II** has a winning strategy.

(5) Using **II**'s strategy, we cover S we a small measurable set, completing the proof.

Recall the following measure-theoretic fact:

Fact (ZF + AC_ω)

For any $A \subseteq \mathbb{R}$ and $\varepsilon > 0$, there exists an open $U \supseteq A$ such that $\mu(U) \leq \mu^*(A) + \varepsilon$.

Taking countable intersections of such open sets, there exists some measurable $E \supseteq A$ such that every measurable subset of $E \setminus A$ is null. Therefore, it suffices to show that:

Under AD, if $S \subseteq \mathbb{R}$ is such that every measurable subset of S is null, then S is null.

With that, we know that $E \setminus A$ is measurable, so $A = E \setminus (E \setminus A)$ is also measurable.

The Covering Game

Fix some $S \subseteq [0,1]$ and $\varepsilon > 0$ such that every measurable subset of S is null. Define:

$$K := \{G \subseteq \mathbb{R} : G \text{ is a finite union of rational intervals}\}$$

Note that K is countable. For each n, we define:

$$K_n := \left\{ G \in K : \mu(G) \leq \frac{\varepsilon}{2^{2(n+1)}} \right\}$$

The covering game is as follows:

I	$a_0 \in \{0,1\}$		$a_1 \in \{0,1\}$		$\cdots$
II		$G_0 \in K_0$		$G_1 \in K_1$	$\cdots$

The *outcome* of the game is the real number defined by:

$$a := \sum_{n=0}^{\infty} \frac{a_n}{2^{n+1}}$$

We ask that player **I** wins iff at the end of the game, the following conditions are fulfilled:

(1) $a \in S$.

(2) $a \notin \bigcup_{n=0}^{\infty} G_n$.

Lemma

*Player **I** does not have a winning strategy in this game.*

Sketch of Proof.

Suppose **I** has some strategy σ. Define:

$$Z := \{x \in S : x \text{ is a possible outcome when } \mathbf{I} \text{ follows } \sigma\}$$

One can show that Z is a measurable subset of S. By the hypothesis on S, Z is null.

Since null sets can be covered by arbitrarily small open sets, we may pick $G_n \in K_n$ such that $Z \subseteq \bigcup_{n=0}^{\infty} G_n$. If **II** plays $(b_0, b_1, \ldots)$, then **I** always lose whenever **I** follows the strategy σ, a contradiction. □

Proof of Theorem.

By AD and the previous lemma, **II** has a winning strategy τ. It suffices to show that $\mu^*(S) \leq \varepsilon$ for arbitrarily $\varepsilon > 0$.

For each $s = (a_0, \ldots, a_{n-1})$ of 0 and 1, let $G_s \in K_n$ be $G_s := \tau(a_0, \ldots, a_{n-1})$, i.e **II** plays G_s if **I** has played $(a_0, \ldots, a_n)$ so far. Since τ is a winning strategy, for any $a = (a_0, a_1, \ldots) \in S$ which **I** plays, we have that $a \in \bigcup_{s \sqsubseteq a} G_s$. Thus:

$$S \subseteq \bigcup_{s \in \{0,1\}^{<\infty}} G_s = \bigcup_{n=1}^{\infty} \bigcup_{s \in \{0,1\}^n} G_s$$

Proof of Theorem (Cont.)

Now for any n, we have that:

$$\mu\left(\bigcup_{s\in\{0,1\}^n} G_s\right) \leq \sum_{s\in\{0,1\}^n} \mu(G_s) \leq 2^n \cdot \frac{\varepsilon}{2^{2n}} = \frac{\varepsilon}{2^n}$$

Therefore:

$$\mu^*(S) \leq \sum_{n=1}^{\infty} \mu\left(\bigcup_{s\in\{0,1\}^n} G_s\right) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$$

as desired. □

Thanks for your attention!

Summary:

(1) Given a set $X \subseteq [\mathbb{N}]^\infty$, in the game G_X, **I** and **II** take turns playing $n_0, n_1, \ldots$, and **I** wins iff $\{n_0, n_1, \ldots\} \in X$.

(2) X is *determined* if either **I** or **II** has a winning strategy in G_X.

(3) AC implies that there exists an undetermined $X \subseteq [\mathbb{N}]^\infty$.

(4) The *Axiom of Determinacy*, AD, is the statement that all $X \subseteq [\mathbb{N}]^\infty$ are determined.

(5) Using the *covering game*, we may show that AD implies that every subset of $\mathbb{R}$ is Lebesgue measurable.