An alternative proof of the Mathias-Silver theorem using the Kastanas game

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Infinite-dimensional Ramsey theory

Infinite-dimensional Ramsey theory started with the study of infinite subsets of natural numbers, $[\mathbb{N}]^\infty$. Let's recall the definition.

Notation

Given $A \in [\mathbb{N}]^\infty$ and $a \in [A]^{<\infty}$, we write:

$$[a, A] := \{B \in [A]^\infty : a \sqsubseteq B\}$$

where $a \sqsubseteq B$ means that $B \cap \max(a) = a$.

Definition

A subset $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is *Ramsey* if for all $A \in [\mathbb{N}]^\infty$ and $a \in [A]^{<\infty}$, there exists some $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}^c$ or $[a, B] \subseteq \mathcal{X}$.

Kastanas (1983) was examining the game-theoretic characterisation of Ramsey sets.

Definition (Kastanas)

Let $A \in [\mathbb{N}]^\infty$, and let $a \in [A]^{<\infty}$. The *Kastanas game* played below $[a, A]$, denoted as $K[a, A]$, is:

I	$A_0 = A$		$A_1 \subseteq B_0$		$A_2 \subseteq B_1$	$\cdots$
II		$x_0 \in A_0$ $B_0 \subseteq A_0$		$x_1 \in A_1$ $B_1 \subseteq A_1$		$\cdots$ $\cdots$

where:

- $\max(a) < x_0 < x_1 < \cdots$.
- A_n, B_n are infinite subsets of $\mathbb{N}$.

The outcome of the game is $a \cup \{x_0, x_1, \dots\} \in [a, A]$.

Definition

We say that **I** (similarly **II**) has a strategy in $K[a, A]$ to *reach* $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ if it has a strategy in $K[a, A]$ to ensure rthe outcome is in $\mathcal{X}$.

Definition

A set $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is *Kastanas Ramsey* if for all $A \in [\mathbb{N}]^\infty$ and $a \in [A]^{<\infty}$, there exists some $B \in [a, A]$ such that one of the following holds:

1. **I** has a strategy in $K[a, B]$ to reach $\mathcal{X}^c$.
2. **II** has a strategy in $K[a, B]$ to reach $\mathcal{X}$.

Definition

We say that **I** (similarly **II**) has a strategy in $K[a, A]$ to *reach* $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ if it has a strategy in $K[a, A]$ to ensure the outcome is in $\mathcal{X}$.

Definition

A set $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is *Kastanas Ramsey* if for all $A \in [\mathbb{N}]^\infty$ and $a \in [A]^{<\infty}$, there exists some $B \in [a, A]$ such that one of the following holds:

1. **I** has a strategy in $K[a, B]$ to reach $\mathcal{X}^c$.
   (Definition of Ramsey: $[a, B] \subseteq \mathcal{X}^c$.)
2. **II** has a strategy in $K[a, B]$ to reach $\mathcal{X}$.
   (Definition of Ramsey: $[a, B] \subseteq \mathcal{X}$.)

Theorem (Kastanas)

A set $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is Ramsey iff Kastanas Ramsey.

There are two well-known descriptive set-theoretic facts about Ramsey subsets of $[\mathbb{N}]^\infty$.

Theorem (Galvin-Prikry)

Every Borel subset of $[\mathbb{N}]^\infty$ is Ramsey.

Theorem (Mathias-Silver)

Every analytic subset of $[\mathbb{N}]^\infty$ is Ramsey.

By the Borel determinacy on $\mathbb{R}^\omega$, the Galvin-Prikry theorem follows from Kastanas' result. However, it is not enough to prove the Mathias-Silver theorem.

Goal. Provide a proof of the Mathias-Silver theorem in the following steps:

1. Define a version of the Kastanas game (and Kastanas Ramsey sets) on $[\mathbb{N}]^\infty \times 2^\infty$. By the Borel determinacy for Polish spaces, all Borel subsets of $[\mathbb{N}]^\infty \times 2^\infty$ are Kastanas Ramsey.
2. Show that Kastanas Ramsey sets are closed under projections. Therefore, analytic subsets of $[\mathbb{N}]^\infty$ are Kastanas Ramsey.
3. By Kastanas' theorem, analytic subsets of $[\mathbb{N}]^\infty$ are Ramsey.

Kastanas game on $[\mathbb{N}]^\infty \times 2^\infty$

Definition

Let $A \in [\mathbb{N}]^\infty$, and let $a \in [\mathbb{N}]^{<\infty}$ and $p \in 2^{|a|}$. The *Kastanas game* played below $[a, A, p]$, denoted as $K[a, A, p]$, is:

I	$A_0 = A$		$A_1 \subseteq B_0$		$\cdots$
II		$x_0 \in A_0$		$x_1 \in A_1$	$\cdots$
		$\varepsilon_0 \in \{0,1\}$		$\varepsilon_1 \in \{0,1\}$	$\cdots$
		$B_0 \subseteq A_0$		$B_1 \subseteq A_1$	$\cdots$

where:

- $\max(a) < x_0 < x_1 < \cdots$.
- A_n, B_n are infinite subsets of $\mathbb{N}$.

The outcome of the game is
$(a \cup \{x_0, x_1, \dots\}, p^\frown(\varepsilon_0, \varepsilon_1, \dots)) \in [a, A] \times 2^\infty$.

Definition

We say that **I** (similarly **II**) has a strategy in $K[a, A, p]$ to *reach* $\mathcal{C} \subseteq [\mathbb{N}]^\infty \times 2^\infty$ if it has a strategy in $K[a, A, p]$ to ensure the outcome is in $\mathcal{C}$.

Definition

A set $\mathcal{C} \subseteq [\mathbb{N}]^\infty \times 2^\infty$ is *Kastanas Ramsey* if for all $A \in [\mathbb{N}]^\infty$, $a \in [A]^{<\infty}$ and $p \in 2^{|a|}$, there exists some $B \in [a, A]$ such that one of the following holds:

1. **I** has a strategy in $K[a, B, p]$ to reach $\mathcal{C}^c$.
2. **II** has a strategy in $K[a, B, p]$ to reach $\mathcal{C}$.

Let $\pi_0 : [\mathbb{N}]^\infty \times 2^\infty \to [\mathbb{N}]^\infty$ be the projection to the first coordinate.

Theorem

If $\mathcal{C} \subseteq [\mathbb{N}]^\infty \times 2^\infty$ is Kastanas Ramsey, then $\pi_0[\mathcal{C}] \subseteq [\mathbb{N}]^\infty$ is Kastanas Ramsey.

We split the proof of the theorem into two lemmas.

Lemma

Let $\mathcal{C} \subseteq [\mathbb{N}]^\infty \times 2^\infty$ be a subset. Let $A \in [\mathbb{N}]^\infty$, $a \in [A]^{<\infty}$. If **II** *has a strategy in $K[a, A, p]$ to reach $\mathcal{C}$ for some $p \in 2^{\mathrm{lh}(a)}$, then* **II** *has a strategy in $K[a, A]$ to reach $\pi_0[\mathcal{C}]$.*

Proof.

The strategy by **II** in the game $K[a, A, p]$ to reach $\mathcal{C}$, with the ε_n's ignored, is a strategy for **II** in $K[a, A]$ to reach $\pi_0[\mathcal{C}]$. □

Lemma

*Let $\mathcal{C} \subseteq [\mathbb{N}]^\infty \times 2^\infty$ be a subset. Let $A \in [\mathbb{N}]^\infty$, $a \in [A]^{<\infty}$. If for all $p \in 2^{\mathrm{lh}(a)}$, there exists some $C \in [a, A]$ such that **I** has a strategy in $K[a, C, p]$ to reach $\mathcal{C}^c$, then there exists some $B \in [a, A]$ such that **I** has a strategy in $K[a, B]$ to reach $\pi_0[\mathcal{C}]^c$.*

Since $\pi_0[\mathcal{C}^c] \neq \pi_0[\mathcal{C}]^c$ in general, the same naive argument doesn't work here.

In the interest of time, we shall prove this lemma only for $a = \emptyset$.

Proof of the second Lemma

Let $B \in [A]^\infty$ and σ be a strategy for **I** in $K[\emptyset, B, \emptyset]$ (in $[\mathbb{N}]^\infty \times 2^\infty$) to reach $\mathcal{C}^c$. How do we define a strategy τ for **I** in $K[\emptyset, B]$ (in $[\mathbb{N}]^\infty$) to reach $\pi_0[\mathcal{C}]^c$?

- Say that the outcome of a complete run in $K[\emptyset, B]$ (in $[\mathbb{N}]^\infty$), following τ, is $D = \{x_0, x_1, \dots\}$.
- $D \notin \pi_0[\mathcal{C}]^c$ iff for all $x \in 2^\infty$, $(D, x) \in \mathcal{C}^c$.
- **Goal.** Design τ such that, for any outcome D and any $x \in 2^\infty$ (in $[\mathbb{N}]^\infty$), there is a simulation of the game in $K[\emptyset, B, \emptyset]$ (in $[\mathbb{N}]^\infty \times 2^\infty$) following σ, such that the outcome is (D, x). By our choice of σ, $(D, x) \in \mathcal{C}^c$.

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$
II	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$
II	

or

I	$A_0 = B$
II	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$

or

I	$A_0 = B$
II	$x_0 \in A_0$ $\varepsilon_0 = 1$

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$

or

I	$A_0 = B$
II	$x_0 \in A_0$ $\varepsilon_0 = 1$

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$
II	$x_0 \in A_0$
	$B_0 \subseteq A_0$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$
II	$x_0 \in A_0$	
	$\varepsilon_0 = 0$	
	$B_0 \subseteq A_0$	

or

I	$A_0 = B$
II	$x_0 \in A_0$
	$\varepsilon_0 = 1$

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	

or

I	$A_0 = B$
II	$x_0 \in A_0$ $\varepsilon_0 = 1$ $A_1^0 \subseteq A_0$

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	
II		$x_0 \in A_0$
		$B_0 \subseteq A_0$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$		$A_1^0 := \sigma(x_0, 0, B_0)$
II		$x_0 \in A_0$	
		$\varepsilon_0 = 0$	
		$B_0 \subseteq A_0$	

or

I	$A_0 = B$		$A_1^1 := \sigma(x_0, 1, A_1^0)$
II		$x_0 \in A_0$	
		$\varepsilon_0 = 1$	
		$A_1^0 \subseteq A_0$	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$	

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	

or

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$
II	$x_0 \in A_0$ $\varepsilon_0 = 1$ $A_1^0 \subseteq A_0$	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$
II	$x_0 \in A_0$	$x_1 \in A_1$
	$B_0 \subseteq A_0$	$B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$
II	$x_0 \in A_0$	
	$\varepsilon_0 := 0$	
	$B_0 \subseteq A_0$	

or

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$
II	$x_0 \in A_0$	
	$\varepsilon_0 = 1$	
	$A_1^0 \subseteq A_0$	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$	$x_1 \in A_1$ $B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 0)$:

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	$x_1 \in A_1^1$ $\varepsilon_1 = 0$

or

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	$x_1 \in A_1^1$ $\varepsilon_1 = 1$

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$	
II		$x_0 \in A_0$ $B_0 \subseteq A_0$	$x_1 \in A_1$ $B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 0)$:

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$		$A_2^0 := \sigma(x_0, 0, B_0, x_1, 0, B_1)$
II		$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	$x_1 \in A_1^1$ $\varepsilon_1 = 0$ $B_1 \subseteq A_1^1$	

or

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$	
II		$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	$x_1 \in A_1^1$ $\varepsilon_1 = 1$

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$	
II		$x_0 \in A_0$ $B_0 \subseteq A_0$	$x_1 \in A_1$ $B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 0)$:

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$		$A_2^0 := \sigma(x_0, 0, B_0, x_1, 0, B_1)$
II		$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	$x_1 \in A_1^1$ $\varepsilon_1 = 0$ $B_1 \subseteq A_1^1$	

or

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$		$A_2^1 := \sigma(x_0, 0, B_0, x_1, 1, A_2^0)$
II		$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	$x_1 \in A_1^1$ $\varepsilon_1 = 1$ $A_2^0 \subseteq A_1^1$	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$
II	$x_0 \in A_0$	$x_1 \in A_1$
	$B_0 \subseteq A_0$	$B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 1)$:

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$
II	$x_0 \in A_0$	$x_1 \in A_1^1$
	$\varepsilon_0 = 1$	$\varepsilon_1 = 0$
	$A_1^0 \subseteq A_0$	$A_2^1 \subseteq A_1^1$

or

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$
II	$x_0 \in A_0$	$x_1 \in A_1^1$
	$\varepsilon_0 = 1$	$\varepsilon_1 = 1$
	$A_1^0 \subseteq A_0$	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$	
II	$x_0 \in A_0$		$x_1 \in A_1$
	$B_0 \subseteq A_0$		$B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 1)$:

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$		$A_2^2 := \sigma(x_0, 1, B_0, x_1, 0, A_2^1)$
II	$x_0 \in A_0$		$x_1 \in A_1^1$	
	$\varepsilon_0 = 1$		$\varepsilon_1 = 0$	
	$A_1^0 \subseteq A_0$		$A_2^1 \subseteq A_1^1$	

or

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$	
II	$x_0 \in A_0$		$x_1 \in A_1^1$
	$\varepsilon_0 = 1$		$\varepsilon_1 = 1$
	$A_1^0 \subseteq A_0$		

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$
II	$x_0 \in A_0$	$x_1 \in A_1$
	$B_0 \subseteq A_0$	$B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 1)$:

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$	$A_2^2 := \sigma(x_0, 1, B_0, x_1, 0, A_2^1)$
II	$x_0 \in A_0$	$x_1 \in A_1^1$	
	$\varepsilon_0 = 1$	$\varepsilon_1 = 0$	
	$A_1^0 \subseteq A_0$	$A_2^1 \subseteq A_1^1$	

or

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$	$A_2^3 := \sigma(x_0, 1, A_1^0, x_1, 1, A_2^2)$
II	$x_0 \in A_0$	$x_1 \in A_1^1$	
	$\varepsilon_0 = 1$	$\varepsilon_1 = 1$	
	$A_1^0 \subseteq A_0$	$A_2^2 \subseteq A_1^1$	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$		$\tau(x_0, B_0, x_1, B_1) := A_2^3$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$		$x_1 \in A_1$ $B_1 \subseteq A_1$	

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 1)$:

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$		$A_2^2 := \sigma(x_0, 1, B_0, x_1, 0, A_2^1)$
II	$x_0 \in A_0$ $\varepsilon_0 = 1$ $A_1^0 \subseteq A_0$		$x_1 \in A_1^1$ $\varepsilon_1 = 0$ $A_2^1 \subseteq A_1^1$	

or

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$		$A_2^3 := \sigma(x_0, 1, A_1^0, x_1, 1, A_2^2)$
II	$x_0 \in A_0$ $\varepsilon_0 = 1$ $A_1^0 \subseteq A_0$		$x_1 \in A_1^1$ $\varepsilon_1 = 1$ $A_2^2 \subseteq A_1^1$	

Abstract Ramsey theory

There were efforts to generalise Ramsey's theory of the reals to other spaces.

Ramsey spaces. Four abstract axioms, **A1**, **A2**, **A3**, **A4**, were developed by Todorčević to capture the Ramsey-theoretic essences of the reals. In particular, $[\mathbb{N}]^\infty$ is an example of a Ramsey space satisfying **A1**-**A4**.

Theorem (Todorčevič)

If $(\mathcal{R}, \leq, r)$ is a closed triple satisfying **A1**-**A4**, *then every analytic subset of $\mathcal{R}$ is Ramsey.*

In 2023, Cano-Di Prisco proposed an abstract version of the Kastanas game for spaces satisfying these axioms.

Theorem (Cano-Di Prisco, 2023)

If $(\mathcal{R}, \leq, r)$ is a closed triple satisfying **A1**-**A4**, *and $\mathcal{R}$ is selective, then $\mathcal{X}$ is Ramsey iff it is Kastanas Ramsey.*

Theorem (Y., 2024)

If $(\mathcal{R}, \leq, r)$ is a closed triple satisfying **A1**-**A4**, *then $\mathcal{X}$ is Ramsey iff it is Kastanas Ramsey.*

Thus, Todorčevič theorem can be translated to "every analytic subset of a Ramsey space is Kastanas Ramsey".

Countable vector spaces. On the other hand, Rosendal studied the Ramsey-theoretic property of *strategically Ramsey subsets* of countable vector spaces. Unlike Todorčevič's framework, countable vector spaces do not satisfy the axioms **A2** (finitisation) and **A4** (pigeonhole).

These sets behave similarly to Ramsey subsets of $[\mathbb{N}]^\infty$ to some extent.

Theorem (Rosendal, 2010)

Let E be a vector space over a countable field of countable dimension. Then every analytic subset of $E^{[\infty]}$ is strategically Ramsey.

Proposition

Let E be a vector space over a countable field of countable dimension. Then $\mathcal{X} \subseteq E^{[\infty]}$ is Kastanas Ramsey iff it is strategically Ramsey.

Thus, Rosendal's theorem can be translated to "every analytic subset of $E^{[\infty]}$ is Kastanas Ramsey".

Question. Is there an overarching theorem that encompasses Todorčevic's theorem and Rosendal's theorem on analytic sets being Kastanas Ramsey.

Weak **A2** spaces

Definition

A triple $(\mathcal{R}, \leq, r)$ is a *weak* **A2** *space*, or just **wA2**-*space*, if it is a closed triple satisfying **A1**, **wA2**, **A3**.

Here, the axiom **wA2** is a weakened version of **A2**, which countable vector spaces satisfy. Thus:

1. If $(\mathcal{R}, \leq, r)$ is a closed triple satisfying **A1**-**A4**, it is a **wA2**-space.
2. If E is a vector space over a countable field, then $E^{[\infty]}$ is a **wA2**-space.

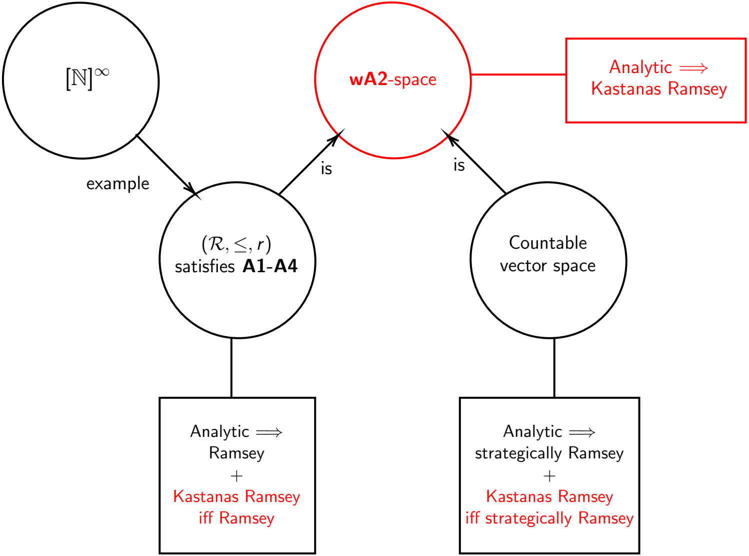
We can apply Cano-Di Prisco's abstract definition of the Kastanas game to **wA2**-spaces.

Theorem (Y., 2024)

Let $(\mathcal{R}, \leq, r)$ be a **wA2***-space.*

1. $(\mathcal{R} \times 2^\infty, \preceq, r')$ *is also a* **wA2***-space, where $\preceq$ and r' are suitably defined.*
2. *If $\mathcal{C} \subseteq \mathcal{R} \times 2^\infty$ is Kastanas Ramsey, then $\pi_0[\mathcal{C}] \subseteq \mathcal{R}$ is Kastanas Ramsey.*
3. *Every analytic subset of $\mathcal{R}$ is Kastanas Ramsey.*

Summary


$[\mathbb{N}]^\infty$
example
$(\mathcal{R}, \leq, r)$ satisfies **A1-A4**
is
wA2-space
is
Countable vector space
Analytic $\Longrightarrow$ Kastanas Ramsey
Analytic $\Longrightarrow$ Ramsey
+
Kastanas Ramsey iff Ramsey
Analytic $\Longrightarrow$ strategically Ramsey
+
Kastanas Ramsey iff strategically Ramsey