Supplementary Notes for Seminar Week 5

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I raised various confusions in this week's seminar, so I am supplementing my usual slides with this set of notes to clarify the confusions I raised.

1 ω-Models

I gave a rather vague definition of what it means for $\mathfrak{A}$ to be an ω-model of KP. Here I give a precise definition. Let ω denote the first countable ordinal in the universe V, and let $\omega^{\mathfrak{A}}$ be the set of $\mathfrak{A}$ such that $\mathfrak{A} \models \omega^{\mathfrak{A}}$ is the first countable ordinal (if it exists).

Definition 1.1. Let $\mathfrak{A}$ be a model of KP. We say that $\mathfrak{A}$ is an ω-model if $\omega^{\mathfrak{A}}$ exists, and there exists a bijective map $j : \omega \to \omega^{\mathfrak{A}}$ such that for all arithmetical formula φ (i.e. formulas in the language of PA), we have that:

$$V \models \varphi(x_1, \ldots, x_n) \iff \mathfrak{A} \models \varphi^{\mathfrak{A}}(j(x_1), \ldots, j(x_n))$$

Here, $\varphi^{\mathfrak{A}}$ denote the arithmetical formula with ω replaced with $\omega^{\mathfrak{A}}$. For instance, if φ is the formula $\forall x \in \omega \, \exists y \in \omega [x < y]$, then $\varphi^{\mathfrak{A}}$ is the formula $\forall x \in \omega^{\mathfrak{A}} \, \exists y \in \omega^{\mathfrak{A}} [x < y]$.

In particular, we do not need to have $\omega \in \mathfrak{A}$. We give an example of an ω-model of ZFC (the same idea applies to models of KP).

Example 1.2. Suppose $(M, \in)$ is a transitive set model of $\mathsf{ZFC} + \exists$a measurable cardinal. Then M is an ω-model, as ω is absolute across transitive models (see Lemma 12.10 of Jech). Let $\mathcal{U} \in M$ be a κ-complete non-principal ultrafilter on κ, and define the *ultrapower* by:

$$V^\kappa/\mathcal{U} := \{[f]_{\mathcal{U}} : f : \kappa \to V \text{ is a function in } V\}$$

where $[f]_\mathcal{U}$ is the equivalence class of functions from κ to V under the equivalence relation:

$$f \sim g \iff \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in \mathcal{U}$$

The membership relation in the ultrapower is defined by:

$$[f]_\mathcal{U} \in^* [g]_\mathcal{U} \iff \{\alpha < \kappa : f(\alpha) \in g(\alpha)\} \in \mathcal{U}$$

It turns out that $(V^\kappa/\mathcal{U}, \in^*)$ is a well-founded model of ZFC. In this case, we have that:

$$\omega^{(V^\kappa/\mathcal{U},\in^*)} = [\alpha \mapsto \omega]_\mathcal{U}$$

where $\alpha \mapsto \omega$ is the constant function mapping all ordinals $< \kappa$ to ω. In particular, we do not literally have that $\omega \in V^\kappa/\mathcal{U}$, but $(V^\kappa/\mathcal{U}, \in^*)$ is still an ω-model as $\omega^{(V^\kappa/\mathcal{U},\in^*)}$ is isomorphic to the true ω. More precisely, we may define $j : \omega \to \omega^{(V^\kappa/\mathcal{U},\in^*)}$ by:

$$j(n) := [\alpha \mapsto n]_\mathcal{U}$$

2 Ordinals in KP Models

2.1 $s(\mathfrak{A})$

If $\mathfrak{A} = (A, E)$ is a model of KP, I defined the following:

$$s(\mathfrak{A}) := \sup\{\text{otp}(S) : \ S \text{ is an initial segment of } \mathbf{ORD}^\mathfrak{A} \wedge S \text{ is well-ordered}\}$$

The author of the book who gave this definition (Weitkamp-Mansfield) justified this definition by asserting that every linear ordering has a maximal well-ordered subset. This definition of “maximal” should not be interpreted as “maximal under inclusions”, for $(\mathbb{Z}, <)$ is a counterexample. I believe the authors meant “maximal under *sections*” - see this MSE post.

I am still not sure why this justifies that $s(\mathfrak{A})$ is well-defined, but we do not need this line of reasoning to do so. We simply note that:

$$\{\text{otp}(S) : S \subseteq \mathbf{ORD}^\mathfrak{A} \wedge (S, E) \text{ is well-ordered}\}$$

is a well-defined subset of ordinals in the universe, so we may take the supremum of this set.

2.2 $s(\mathfrak{A})$ of non-ω-Models

This is a small clarification on my example that if $\mathfrak{A}$ is not an ω-model, then $s(\mathfrak{A}) = \omega$. I said that this is because $\omega \notin \mathfrak{A}$. To be more precise, I should have said that it is because $\omega^\mathfrak{A}$ does not exist.

2.3 Bounding $s(\mathfrak{A})$

Another small clarification: If α is some ordinal, when I say that $\alpha \leq s(\mathfrak{A})$, I more precisely mean that we may define a well-ordered set S of ordinals in $\mathfrak{A}$ such that, in the universe, $\mathrm{otp}(S) \geq \alpha$. We do not literally mean that $\alpha \subseteq \mathfrak{A}$.