NUS Reading Seminar Summer 2023

Session 3

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Trees

Recall that an r-sequential tree is a subset:

$$T \subseteq \mathrm{Seq}_r = \{(s_1, \ldots, s_r) \in (\omega^\omega)^r : |s_1| = \cdots = |s_r|\}$$

that is closed under initial segments - i.e. if $(s_1, \ldots, s_r) \in T$, then for all $n \leq |s_i|$, $(s_1 \restriction n, \ldots, s_r \restriction n) \in T$.

We now consider a slight generalisation of such trees. We define $\mathrm{Seq}(K) := K^{<\omega}$.

Definition

Let K be a set and $r \geq 1$. A tree on $\omega^r \times K$ is a subset $T \subseteq \mathrm{Seq}_r \times \mathrm{Seq}(K)$ that is closed under initial segments.

For instance, an r-dimensional sequential tree is a tree on ω^r.

Given a tree T on $\omega^r \times K$, for $x \in \omega^r$ we can then once again define the "projection" as:

$$T(x) := \{h \in \mathrm{Seq}(K) : (x \restriction |h|, h) \in T\}$$

A recap of Π^1_1 normal form:

Theorem (Normal Form for Π^1_1 Sets)

Let $A \subseteq \omega^\omega$. Then A is $\Sigma^1_1(a)$ iff there exists a tree $T \subseteq \mathrm{Seq}_2$ recursive in a such that:

$$x \in A \iff T(x) \text{ is ill-founded}$$

In other words, we have that:

$$A = \{x \in \omega^\omega : T(x) \text{ is ill-founded}\}$$

Notation

Let T be a tree on $\omega \times K$. Then:

$$p[T] := \{x \in \omega^\omega : T(x) \text{ is ill-founded}\}$$

Definition

Let κ be an infinite cardinal. A set $A \subseteq \omega^\omega$ is *κ-Suslin* if $A = p[T]$ for some tree T on $\omega \times \kappa$.

Therefore, we may reword Π^1_1 normal form theorem to say that:

$A \subseteq \omega^\omega$ is $\Sigma^1_1(a)$ iff $A = p[T]$ for some tree T on $\omega \times \omega$ recursive in a.

$\mathbf{\Sigma}^1_2$-Sets

The main theorem of this section is as follows.

Theorem

If $A \subseteq \omega^\omega$ is $\Sigma^1_2(a)$, then $A = p[T]$ for some tree T on $\omega \times \omega_1$ such that $T \in L[a]$.

Loosely speaking, $T \in L[a]$ means that T can be defined from a and some really simple objects.

Similar to the proof of Π^1_1 normal forms, we shall try to find an appropriate relation recursive in a, then "close it under initial segments".

In other words, we wish to find some tree T on $\omega \times \omega_1$, with $T \in L[a]$, such that:

$$x \in A \iff \exists h \in \omega_1^\omega \, \forall n \, (x{\restriction}n, h{\restriction}n) \in T$$

The proof of the theorem would follow the steps below:

(1) "Simplify" what it means for A to be $\Sigma^1_2(a)$.

(2) Find a tree T' on $\omega^2 \times \omega_1$, with $T \in L[a]$, such that:

$$x \in A \iff \exists y \in \omega^\omega \, \exists h \in \omega_1^\omega \, \forall n \, (x{\restriction}n, y{\restriction}n, h{\restriction}n) \in T'$$

(3) Transform T' into a tree T on $\omega \times \omega_1$ with the desired property.

Proof, Step 1.

Let $A \subseteq \omega^\omega$ be $\Sigma^1_2(a)$. In other words, there exists a $\Pi^1_1(a)$-set $B \subseteq (\omega^\omega)^2$ such that:

$$x \in A \iff \exists y \in \omega^\omega \, (x, y) \in B$$

By the Π^1_1 normal form, there exists some tree $U \subseteq \mathrm{Seq}_3$ recursive in a such that:

$$\begin{aligned}
& x \in A \\
\iff & \exists y \in \omega^\omega \, U(x, y) \text{ is well-founded} \\
\iff & \exists y \in \omega^\omega \, \exists \text{a rank function } f : U(x, y) \to \omega_1 \\
\iff & \exists y \in \omega^\omega \, \exists f : \mathrm{Seq} \to \omega_1 \text{ s.t. } f \restriction U(x, y) \text{ is order-preserving}
\end{aligned}$$

Note that by the countability of $U(x, y)$, we assumed that $\mathrm{ran}(f) \subseteq \omega_1$.

Proof, Step 1. (Cont.)

Fix some recursive enumeration $\mathrm{Seq} = \{u_n : n < \omega\}$ such that $|u_n| \leq n$ for all n. Given a function f with $\mathrm{dom}(f) \subseteq \omega$, we define f with $\mathrm{dom}(f^*) \subseteq \mathrm{Seq}$ by $f^*(u_n) := f(n)$. Using this enumeration we get that:

$$x \in A \iff \exists y \in \omega^\omega \, \exists h \in \omega_1^\omega \text{ s.t. } h^* \restriction U(x, y) \text{ is order-preserving}$$

Proof, Step 2.

We define a tree T' on $\omega^2 \times \omega_1$ by "closing" the relation in the previous slide under initial segments. More precisely, stipulate that:

$$(s, t, h) \in T' \iff h^* \upharpoonright U_{s,t} \text{ is order-preserving}$$

where:

$$U_{s,t} := \{u \in \mathrm{Seq} : |u| \leq |s| \wedge (s \upharpoonright |u|, t \upharpoonright |u|, u) \in U\}$$

It's easy to check that T' is a tree.

Proof, Step 2. (Cont.)

Now observe that given $x, y \in \omega^\omega$, we have:

$$\begin{aligned} U(x,y) &= \{u \in \mathrm{Seq} : (x\restriction|u|, y\restriction|u|, u) \in U\} \\ &= \left\{u \in \mathrm{Seq} : u \in U_{x\restriction|u|, y\restriction|u|}\right\} \\ &= \bigcup_{n<\omega} \{u \in \mathrm{Seq} : |u| \le n \wedge u \in U_{x\restriction n, y\restriction n}\} \\ &= \bigcup_{n<\omega} U_{x\restriction n, y\restriction n} \end{aligned}$$

Therefore, given $x, y \in \omega^\omega$ and $h \in \omega_1^\omega$, we have that:

$$\begin{aligned} h \in T'(x,y) &\iff \forall n\,(x\restriction n, y\restriction n, h\restriction n) \in T' \\ &\iff \forall n\,(h\restriction n)^*\restriction U_{x\restriction n, y\restriction n} \text{ is order-preserving} \\ &\iff h^*\restriction U(x,y) \text{ is order-preserving} \end{aligned}$$

Proof, Step 2. (Cont.)

Therefore:

$$\begin{aligned} x \in A &\iff \exists y \in \omega^\omega \, \exists h \in \omega_1^\omega \text{ s.t. } h^* \restriction U(x,y) \text{ is order-preserving} \\ &\iff \exists y \in \omega^\omega \, \exists h \in \omega_1^\omega \, h \in T'(x,y) \\ &\iff \exists y \in \omega^\omega \, \exists h \in \omega_1^\omega \, \forall n \, (x\restriction n, y\restriction n, h\restriction n) \in T' \end{aligned}$$

Hence this T' is the desired tree for step (2).

Proof, Step 3.

We first transform the tree T' (on $\omega^2 \times \omega_1$) into a tree T'' on $\omega \times (\omega \times \omega_1)$ by the map:

$$((s(0), \dots, s(n-1)), (t(0), \dots, t(n-1)), (h(0), \dots, h(n-1)))$$
$$\downarrow$$
$$((s(0), \dots, s(n-1)), ((t(0), h(0)), \dots, (t(n-1), h(n-1)))$$

Clearly this map is recursive. This gives us:

$$x \in A \iff \exists g \in (\omega \times \omega_1)^\omega \, \forall n \, (x{\restriction}n, g{\restriction}n) \in T''$$

Using a definable correspondence between ω_1 and $\omega \times \omega_1$, we get a tree T such that:

$$x \in A \iff \exists g \in \omega_1^\omega \, \forall n \, (x{\restriction}n, g{\restriction}n) \in T$$

so $A = p[T]$. Clearly T is constructible from a.

We remark that if $x \in A$, so $T(x)$ is ill-founded, then by reversing the proof we have an algorithm which obtains a real $y \in \omega^\omega$ (dependent only on x and ω_1) such that $U(x, y)$ is well-founded. This will be important in proving Shoenfield absoluteness theorem later.

Shoenfield Absoluteness Theorem

Theorem

If P is a $\Sigma^1_2(a)$ relation, then P is absolute for every inner models M of $\mathsf{ZF} + \mathsf{DC}$ such that $a \in M$. In particular, P is absolute for L.

One may think that we can mimic the proof of Mostowski absoluteness theorem to prove Shoenfield absoluteness theorem. However, this does not work.

Suppose P is $\Sigma^1_1(a)$ and $R \subseteq \mathrm{Seq}_2$ is a recursive relation in which:

$$P(x) \iff \exists y \in \omega^\omega \, \forall n \, R(x{\restriction}n, y{\restriction}n)$$

We defined the tree $T \subseteq \mathrm{Seq}_2$ by:

$$T := \{(s,t) \in \mathrm{Seq}_2 : \forall n \leq |s| \, R(s{\restriction}n, t{\restriction}n)\}$$

then showed that:

$$P(x) \iff T(x) \text{ is ill-founded}$$

We proved Mostowski absoluteness theorem as follows:

(1) If $M \models P(x)$, then $M \models T(x)$ is ill-founded, so $[T(x)] \neq \emptyset$.
(2) If $M \models \neg P(x)$, then $M \models T(x)$ is well-founded, so there exists a rank function on T.

We implicitly used the fact that the tree T, constructed in V and in M, **are the same**.

What about the tree T constructed such that $P = p[T]$ when P is $\Sigma^1_2(a)$?

We started with:

$$P(x) \iff \exists y \in \omega^\omega\, U(x, y) \text{ is well-founded}$$

where $U \subseteq \mathrm{Seq}_3$, and constructed a tree T on $\omega \times \omega_1$ such that from U. We immediately see that the tree T constructed in M need not be the same as that in V - for instance, we need not have $\omega_1^M = \omega_1$.

We thus have to work around this issue when proving Shoenfield absoluteness theorem.

Proof.

Suppose P is $\Sigma^1_2(a)$. As discussed before, there exists a tree $U \subseteq \mathrm{Seq}_3$, recursive in a, such that:

$$P(x) \iff \exists y\, U(x, y) \text{ is well-founded}$$

This U is independent of the choice of models, i.e. we also have that:

$$M \models P(x) \iff \exists y \in M\, M \models U(x, y) \text{ is well-founded}$$

For any relation R on $\omega^{<\omega}$, the statement "R is well-founded" is Π^1_1 (Exercise), so it is absolute by Mostowski absoluteness theorem. Therefore:

$$M \models P(x) \iff \exists y \in M\, U(x, y) \text{ is well-founded}$$

This immediately proves that if $M \models P(x)$, then $P(x)$ holds. It remains to show the converse.

Proof (Cont.)

Suppose $P(x)$ holds. Let T be the tree on $\omega \times \omega_1$, constructed from U in V, such that $P = p[T]$. Therefore:

$$T(x) \text{ is ill-founded}$$

Since well-foundedness is absolute, we have that:

$$M \models T(x) \text{ is ill-founded}$$

Despite the fact that $T \in M$, we need not have $M \models P = p[T]$. However, as remarked earlier, we can instead reverse the proof of that P is ω_1-Suslin to obtain a $y \in (\omega^\omega)^M$ such that:

$$M \models U(x, y) \text{ is well-founded}$$

Hence $M \models P(x)$. □

DC is used here for the fact that "R is well-founded" is a Π^1_1 statement. For more details, see Lemma 25.9 of Jech.

However, all trees involved can in fact be canonically well-ordered, as they are subsets of $\omega^{<\omega}$. Consequently, we do not require DC to choose an infinite branch when proving that "R is well-founded" is Π^1_1. Therefore, Shoenfield absoluteness theorem applies to models of ZF, and its inner models of ZF.

A few concluding remarks:

(1) Given $x \subseteq \omega$, we say that x is $\Sigma^1_n(a)$ (resp. $\Pi^1_n(a)$) if the set $\{e_x\}$, where e_x is the indicator function of the set x, is $\Sigma^1_n(a)$ (resp. $\Pi^1_n(a)$). Shoenfield absoluteness theorem implies that if x is $\Sigma^1_2(a)$ or $\Pi^1_2(a)$, then $x \in L[a]$. In particular, every Σ^1_2/Π^1_2 real is constructible.

(2) There exists a model of set theory (without assuming large cardinals) in which there is a non-constructible Δ^1_3 real. Thus, Shoenfield absoluteness theorem is the best possible ZFC absoluteness theorem.

The power of Shoenfield absoluteness lies in the following result.

Corollary

If P is a $\mathbf{\Sigma}^1_2/\mathbf{\Pi}^1_2$ statement, and $\mathrm{ZFC} \vdash P$, then $\mathrm{ZF} \vdash P$.

Proof.

Let M be a model of ZF. Then L^M is a model of ZFC. Since $\mathrm{ZFC} \vdash P$, $L^M \models P$. By Shoenfield absoluteness theorem, $M \models P$. Since this holds for any model of ZF, by Gödel's completeness theorem, $\mathrm{ZF} \vdash P$. □

Many statements in "ordinary mathematics" are "simple enough" to be of complexity Σ^1_2/Π^1_2 or lower. Examples include:

(1) Brouwer fixed point theorem.
(2) Hanh-Banach theorem for separable spaces.
(3) The existence of algebraic closures for countable fields.

See more examples here.