NUS Reading Seminar Summer 2023

Session 4

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Silver Indiscernibles

We begin by introducing indiscernibles, a concept in set theory with deep relations to various large cardinal axioms.

Definition

Let I be a linearly ordered subset of a model $\mathfrak{A}$. We say that I is *indiscernible over* $\mathfrak{A}$ iff for every parameter free formula $\phi(x_1, \dots, x_n)$, and every two increasing sequences $(\alpha_1, \dots, \alpha_n)$, $(\beta_1, \dots, \beta_n)$ in I, we have that:

$$\mathfrak{A} \models \phi[\alpha_1, \dots, \alpha_n] \iff \mathfrak{A} \models \phi[\beta_1, \dots, \beta_n]$$

Theorem (Silver)

If there exists a Ramsey cardinal, then:

1. *If $\kappa < \lambda$ are uncountable cardinals, then $(L_\kappa, \in) \preceq (L_\lambda, \in)$.*
2. *The is a closed unbounded class of ordinals I, containing all uncountable cardinals, such that for every uncountable cardinal κ:*
   - $|I \cap \kappa| = \kappa$.
   - *$I \cap \kappa$ is indiscernible over $(L_\kappa, \in)$.*
   - *Every $a \in L_\kappa$ is definable in $(L_\kappa, \in)$ from $I \cap \kappa$.*

For instance, a measurable cardinal is a Ramsey cardinal. The class I is called the class of *Silver indiscernibles*.

Lemma

If the consequence of Silver's theorem holds, then for all uncountable cardinals κ, $(L_\kappa, \in) \preceq (L, \in)$.

Proof.

We wish to show that for all formulas φ and $a_1, \ldots, a_n \in L_\kappa$, we have that:

$$L \models \varphi[a_1, \ldots, a_n] \implies L_\kappa \models \varphi[a_1, \ldots, a_n]$$

Fix a formula φ, and suppose $L \models \varphi[a_1, \ldots, a_n]$. By reflection principle, there exists some $\lambda > \kappa$ such that $L_\lambda \models \varphi[a_1, \ldots, a_n]$. Since $(L_\kappa, \in) \preceq (L_\lambda, \in)$, $L_\kappa \models \varphi[a_1, \ldots, a_n]$. □

$0^\sharp$

$0^\sharp$ is a subset of ω which encodes exactly all formulas which L satisfies.

Let $a \in L$, so $a \in L_\kappa$ for some uncountable cardinal κ. Then there exists a formula ϕ and $\alpha_1 < \cdots < \alpha_n$ in $I \cap \kappa$ such that:

$$L_\kappa \models a \text{ is the unique } x \text{ which } \phi(x, \alpha_1, \ldots, \alpha_n) \text{ holds}$$

This means that $a \in L_\kappa$ is "witnessed" by the fact that $L_\kappa \models \varphi[\alpha_1, \ldots, \alpha_n]$, where:

$$\varphi[\gamma_1, \ldots, \gamma_n] \leftrightarrow \exists! x\, \phi(x, \gamma_1, \ldots, \gamma_n)$$

Thus we may just focus on formulas which take in indiscernibles as parameters.

Since $I \cap \kappa$ is indiscernible over L_κ, if $\kappa \geq \aleph_\omega$, then:

$$\begin{aligned} L \models \varphi[\alpha_1, \ldots, \alpha_n] &\iff L_\kappa \models \varphi[\alpha_1, \ldots, \alpha_n] \\ &\iff L_\kappa \models \varphi[\aleph_1, \ldots, \aleph_n] \\ &\iff L_{\aleph_\omega} \models \varphi[\aleph_1, \ldots, \aleph_n] \end{aligned}$$

Definition

$$0^\sharp = \{\ulcorner \varphi \urcorner : L_{\aleph_\omega} \models \varphi[\aleph_1, \ldots, \aleph_n]\}$$

We abbreviate the conclusion of Silver's theorem as "$0^\sharp$ exists".

We discuss some consequences of the existence of $0^\sharp$.

Fact

If $0^\sharp$ exists, then every uncountable cardinal is inaccessible in L. In particular, $V \neq L$.

Proof.

Since $L \models \aleph_1$ is regular, $L \models \aleph_\alpha$ is regular for all $\alpha \geq 1$ by indiscernibility. Similarly, $L \models \aleph_\omega$ is a limit cardinal, so $L \models \aleph_\alpha$ is a limit cardinal for all $\alpha \geq 1$ by indiscernibility. $\square$

Fact

If $0^\sharp$ exists, then $|V_\alpha \cap L| \leq |\alpha|$. In particular, $\mathcal{P}(\omega) \cap L$ is countable.

See Corollary 18.5 of Jech for a proof.

Analytic Determinacy

The existence of $0^\sharp$ has a game-theoretic formulation.

Theorem (Martin, Harrington)

In ZFC*, the following are equivalent:*

1. $0^\sharp$ *exists.*
2. *Analytic determinacy* (Σ^1_1-AD)*, i.e. every* Σ^1_1 *set is determined.*

We first the easier direction $0^\sharp \to \Sigma^1_1$-AD, and leave the converse for another day. We fix some Π^1_1-set $A \subseteq \omega^\omega$. By Π^1_1 normal form theorem, there exists a recursive tree T on ω^2 such that:

$$x \in A \iff T(x) \text{ is well-founded}$$

We consider the following attempt of a proof that Σ^1_1-AD holds (in just ZFC).

Consider the game G where Player I chooses an integer n_{2k}, and Player II responds by choosing $(n_{2k+1}, m_{2k}, m_{2k+1})$, where they are all integers.

Turn	I	II
1	n_0	(n_1, m_0, m_1)
2	n_2	(n_3, m_2, m_3)
$\vdots$	$\vdots$	$\vdots$

Define two reals $x(k) := n_k$ and $y(k) := m_k$. We assert that Player II wins iff $(x, y) \in [T]$.

Since $[T]$ is a closed subset of ω^ω, it is determined by open determinacy.

If II has a winning strategy τ, then II can win G_A by playing G_A as if II is playing G. In other words, define the strategy σ by:

$$\sigma(n_0, n_2, \ldots, n_{2k}) = n_{2k+1},$$
$$\text{where } \tau(n_0, n_2, \ldots, n_{2k}) = (n_{2k+1}, m_{2k}, m_{2k+1})$$

Since τ is a winning strategy for II, $(x, y) \in [T]$. Then $y \in T(x)$, so $T(x)$ is ill-founded, hence $x \notin A$.

What if I has a winning strategy τ? We cannot use the same argument above, since, for instance, $\tau(n_1, m_0, m_1)$ may be different from $\tau(n_1, m_0', m_1')$ if $(m_0, m_1) \neq (m_0', m_1')$.

The idea using $0^\sharp$ is to modify G as follows: Instead of playing $m_0, m_1, \ldots$, we require II to play uncountable cardinals $\xi_0, \xi_1, \ldots$. We then use the indiscernibility of uncountable cardinals to patch the gap in the argument above.

Definition

The *Kleene-Brouwer (KB) ordering* is the ordering $\preceq$ on $\omega^{<\omega}$ as follows: For $s, t \in \omega^{<\omega}$, we have that $s \preceq t$ iff $t \sqsubseteq s$, or if k is the least integer such that $s(k) \neq t(k)$, then $s(k) < t(k)$.

Note that the KB ordering is a linear order on $\omega^{<\omega}$.

Lemma

A tree T is well-founded iff it is well-ordered by the KB well-ordering.

Sketch of Proof.

If T is ill-founded, then $s_0 \sqsubseteq s_1 \sqsubseteq \cdots$ gives us a $\preceq$-decreasing sequence.

If $s_0 \succeq s_1 \succeq \cdots$, then we obtain a branch on T as follows:

1. We have that $s_0(0) \geq s_1(0) \geq \cdots$ (ignoring the finitely many strings which $|s_n| = 0$). Let $k_0 := \min\{s_n(0) : n < \omega\}$, and let n_0 be the least integer such that $s_{n_0}(0) = k_0$.
2. We have that $s_{n_0}(1) \geq s_{n_0+1}(1) \geq \cdots$ (ignoring the finitely many strings which $|s_n| \leq 1$). Let $k_1 := \min\{s_n(1) : n \geq n_0\}$, and let n_1 be the least integer such that $s_{n_1}(1) = k_1$.
3. Repeat to get a infinite sequence $(k_0, k_1, k_2, \dots) \in [T]$.

$\square$

We now instead fix some Σ^1_1-set $A \subseteq \omega^\omega$. By Π^1_1 normal form theorem, there exists a recursive tree T on ω^2 such that:

$$x \notin A \iff T(x) \text{ is well-founded}$$

We fix an enumeration $\mathrm{Seq} = \{t_n : n < \omega\}$. Given $s \in \omega^{<\omega}$, we introduce the notation:

$$T_s := \{t_n \in \omega^{<\omega} : n < |s| \wedge (s \restriction |t_n|, t_n) \in T\}$$

Note that $T(x) = \bigcup_{n<\omega} T_{x\restriction n}$ (a similar fact was proven last week).

Let $\kappa := \aleph_{\aleph_1}$ - it is chosen so that every countable well-order can be embedded to the set of cardinals below κ.

Definition

Let $u \in \kappa^{<\omega}$, and let $s \in \omega^{<\omega}$. We say that u *respects* $(T_s, \prec)$ if $|u| = |s|$, and that for all $i, j < |u|$:

1. If $t_i \notin T_s$, then $u(i) = 0$.
2. If $t_i, t_j \in T_s$ and $t_i \prec t_j$, then $u(i) < u(j)$.

Here, by $u(t)$ we mean $u(n)$, where t is the n^{th} element of T_s under $\prec$.

Definition

Let $h \in \kappa^\omega$, and let $T' \subseteq \omega^{<\omega}$ be a tree. We say that h respects $(T, \prec)$ if for all i, j:

1. If $t_i \notin T$, then $u(i) = 0$.
2. If $t_i, t_j \in T$ and $t_i \prec t_j$, then $u(i) < u(j)$.

Now define a tree U on $\omega \times \kappa$ as follows:

$$U := \{(s, u) : u \text{ respects } (T_s, \prec)\}$$

Since T is recursive (hence $T \in L$), $U \in L$. We see that:

$$\begin{aligned}
x \notin A &\iff T(x) \text{ is well-founded} \\
&\iff T(x) \text{ is } \preceq\text{-well-ordered} \\
&\iff \exists h \in \kappa^\omega \; h \text{ respects } (T(x), \prec) \\
&\iff \exists h \in \kappa^\omega \, \forall n \; h{\restriction}n \text{ respects } (T_{x\restriction n}, \prec) \\
&\iff \exists h \in \kappa^\omega \, \forall n \, (x{\restriction}n, h{\restriction}n) \in U \\
&\iff U(x) \text{ is ill-founded}
\end{aligned}$$

We shall consider a game G' concerning U, and shows that if G' is determined, then so is the game G_A.

In G', Player I chooses an integer n_{2k}, and Player II responds by choosing $(n_{2k+1}, \xi_{2k}, \xi_{2k+1})$, where n_{2k+1} is an integer and ξ_{2k}, ξ_{2k+1} are ordinals $< \kappa$.

Turn	I	II
1	n_0	(n_1, ξ_0, ξ_1)
2	n_2	(n_3, ξ_2, ξ_3)
⋮	⋮	⋮

Let $x(k) = n_k$ and $h(k) = \xi_k$. We assert that Player II wins iff (x, h) is a branch of U.

Lemma

G' is determined.

Proof.

The proof is basically the same as that of open determinacy, but for uncountable trees. Suppose I has no winning strategy.

1. For any n_0 that I plays, II can play some (n_1, ξ_0, ξ_1) such that I has not lost yet.
2. Similarly, on the k^{th} turn, II can play some $(n_{2k+1}, \xi_{2k}, \xi_{2k+1})$ such that II has not lost yet, regardless of what I responds with on the previous turn.

This is a winning strategy for II - if $(x, h) \in U$, then II must have played outside a branch somewhere in the middle of the game. $\square$

Write the game G_A in the following manner:

Turn	I	II
1	n_0	n_1
2	n_2	n_3
$\vdots$	$\vdots$	$\vdots$

We have seen earlier that if **II has a winning strategy for** G', then II can play G_A as if II is playing G' and win.

If I has a winning strategy τ, then things are more complicated. Note that we can ensure that $\tau \in L$ (by examining the proof of open determinacy). We first introduce a notation.

Fix some $s \in \omega^{<\omega}$, and let $z \subseteq \kappa$ such that $|z| = |T_s|$. Let $u \in \kappa^{<\omega}$ be the (unique) sequence such that $\mathrm{ran}(u) \subseteq \{0\} \cup z$ and u respects $(T_s, \prec)$ (u sends the n^{th} element of T^s to the n^{th} ordinal in z). Thus, we define:

$$\tau[s, z] := \tau(s, u)$$

where $\tau(s, u)$ represents the integer that I should play when s and u have been played so far in the game G'.

We now define a strategy σ for I in the game G_A as follows: If $s \in \omega^{<\omega}$ has been played so far, then:

$$\sigma(s) := \tau[s, \{\aleph_1, \aleph_2, \ldots, \aleph_{|T_s|}\}]$$

Clearly $\sigma \in L$.

Claim

σ is a winning strategy for I in the game G_A.

Proof.

Suppose not. Let $x := \{n_0, n_1, n_2, \dots\} \notin A$ be the real played in G_A, where $n_0, n_2, \dots$ are played according to the strategy σ by I. Then $T(x)$ is well-founded, so there exists a KB order-preserving map:

$$h : T(x) \to \{\lambda < \kappa : \lambda \text{ is an uncountable cardinal}\}$$

In other words, $(x, h) \in [U]$.

Proof (Cont.)

Now consider I and II playing the game G' by stipulating that:

Turn	I	II
1	$x(0)$	$(x(1), h(0), h(1))$
2	$x(2)$	$(x(3), h(2), h(3))$
$\vdots$	$\vdots$	$\vdots$

We shall show that for all n, $\sigma(x\restriction 2n) = \tau(x\restriction 2n, h\restriction 2n)$. This gives us the required contradiction, as τ being a winning strategy implies that $(x, h) \notin [U]$.

Proof (Cont.)

Let $z := \mathrm{ran}(h{\restriction}2n)$, which is a finite subset of κ. Note that $\tau[x{\restriction}2n, z] = \tau(h{\restriction}2n, h{\restriction}2n)$. We enumerate $z = \{\lambda_1, \dots, \lambda_k\}$ in an increasing manner. By the indiscernibility of uncountable cardinals:

$$\begin{aligned}\tau(x{\restriction}2n, h{\restriction}2n) &= \tau[x{\restriction}2n, \{\lambda_1, \dots, \lambda_k\}] \\ &= \tau[x{\restriction}2n, \{\aleph_1, \dots, \aleph_k\}] \\ &= \sigma(x{\restriction}2n)\end{aligned}$$

□

The converse is more difficult to prove, and uses admissible set theory and some recursion theory. It uses the following well-known deep result in set theory:

Theorem (Kunen)

$0^\sharp$ *exists iff there exists a non-trivial elementary embedding* $j : L \to L$.

In other words, for the converse we show that if Σ^1_1-AD holds, then we can construct a non-trivial elementary embedding $j : L \to L$.