

NUS Reading Seminar Summer 2023

Session 5

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Kripke-Platek Set Theory

Kripke-Platek (usually abbreviated as KP) set theory is a weakened version of ZF set theory. It is closely related to descriptive set theory and higher recursion theory.

The Axiomatic system KP is the set of the following axioms:

1. **Extensionality, Pairing and Union.**
2. **Regularity:** For every formula $\phi(x)$ is a formula with y not occurring free, then:

$$\exists x \phi(x) \rightarrow \exists x[\phi(x) \wedge \forall y \in x \neg \phi(y)]$$

3. **Δ_0 -Separation:** For each Δ_0 -formula ϕ :

$$\forall x \exists y \forall z [z \in y \leftrightarrow (z \in x \wedge \phi(z))]$$

4. **Δ_0 -Collection:** For each Δ_0 -formula ϕ :

$$\forall x [(\forall y \in x \exists z \phi(y, z)) \rightarrow \exists u \forall y \in x \exists z \in u \phi(y, z)]$$

KP_ω is used to denote $KP +$ Axiom of infinity (in the usual ZF sense).

Admissible set theory thus studies models of KP. These models need not be ω -models - that is, while a model \mathfrak{A} of KP may contain a set $x \in \mathfrak{A}$ such that:

$$\mathfrak{A} \models "(x, \in) \models \text{PA}"$$

we need not have $x = \omega$ (where ω is taken in the universe). In other words, (x, \in) is a *non-standard model of arithmetic*.

If $\omega \in \mathfrak{A}$, then \mathfrak{A} has a standard model of arithmetic, so we say that \mathfrak{A} is an ω -model. Note that if $\mathfrak{A} \models \text{KP}$ and \mathfrak{A} is well-founded, then \mathfrak{A} is an ω -model - its transitive collapse has a copy of ω .

A model being an ω -model is helpful as:

Lemma

If \mathfrak{A} is an ω -model of KP, $x \in \mathfrak{A}$ and $\phi(x)$ is an arithmetical formula, then:

$$\phi(x) \iff \mathfrak{A} \models \phi(x)$$

Δ_1 Properties of KP

KP models in fact satisfy a stronger variant of separation axiom.

Proposition

$KP \vdash \Delta_1$ -separation. That is, if φ is a Δ_1 -formula, then:

$$KP \vdash \forall x \exists y \forall z [z \in y \leftrightarrow (z \in x \wedge \varphi(z))]$$

Proof.

Work in a model of KP. We wish to show that for any set x , there exists some set y such that:

$$\forall z [z \in y \leftrightarrow (z \in x \wedge \varphi(z))]$$

Let ϕ, ψ be Δ_0 -formulas such that for all z , $\varphi(z)$ iff $\exists w \phi(z, w)$ iff $\neg \exists w \psi(z, w)$. In other words, we have that:

$$\forall z [\exists w \phi(z, w) \leftrightarrow \neg \exists w \psi(z, w)]$$

Taking \rightarrow in particular, we have that:

$$\forall z \exists w [\phi(z, w) \vee \psi(z, w)]$$

Proof (Cont.)

Then in particular:

$$\forall z \in x \exists w [\phi(z, w) \vee \psi(z, w)]$$

Applying Δ_0 -collection, we obtain some u such that:

$$\forall z \in x \exists w \in u [\phi(z, w) \vee \psi(z, w)]$$

Now let:

$$y := \{z \in x : \exists w \in u \phi(z, w)\}$$

This is a well-defined set as “ $\exists w \in u \phi(z, w)$ ” is Δ_0 , and we have Δ_0 -separation. Then this set y works. \square

Proposition

$KP \vdash \Sigma_1$ -collection. *That is, if ϕ is a Σ_1 -formula, then:*

$$KP \vdash \forall x[(\forall y \in x \exists z \phi(y, z)) \rightarrow \exists u \forall y \in x \exists z \in u \phi(y, z)]$$

Proof.

Suppose $\varphi(y, z)$ iff $\exists w \phi(y, z, w)$, where ϕ is Δ_0 . Fix some set x , and suppose:

$$\forall y \in x \exists z \exists w \phi(y, z, w)$$

Define a formula ψ by stipulating that:

$$\psi(y, v) \iff v = \{z, w\} \wedge \phi(y, z, w)$$

This is Δ_0 , as “ $v = \{z, w\}$ ” is Δ_0 (and it makes sense as KP has the pairing axiom). Then the first statement implies that:

$$\forall y \in x \exists u \psi(y, u)$$

Proof (Cont.)

By Δ_0 -Collection, there exists a set u such that:

$$\begin{aligned} & \forall y \in x \exists v \in u \psi(y, v) \\ \implies & \forall y \in x \exists v \in u [v = \{z, w\} \wedge \phi(y, z, w)] \\ \implies & \forall y \in x \exists z \in \bigcup u \exists w \phi(y, z, w) \\ \implies & \forall y \in x \exists z \in \bigcup u \varphi(y, z) \end{aligned}$$

so collection holds for φ .



Ordinals in KP models

If \mathfrak{A} is a transitive model of KP, then $\mathfrak{A} \cap \mathbf{ORD}$ is an initial segment of \mathbf{ORD} , and is hence an ordinal. However, \mathfrak{A} need not even be well-founded (the ordinals in \mathfrak{A} need not even be well-ordered!), so we need to adopt a different notion of “ $\mathfrak{A} \cap \mathbf{ORD}$ ”.

Notation

Let $\mathfrak{A} \models \text{KP}$. Then:

$$s(\mathfrak{A}) := \sup\{\text{otp}(S) : S \text{ is an initial segment of } \mathbf{ORD}^{\mathfrak{A}} \\ \wedge S \text{ is well-ordered}\}$$

Example

If \mathfrak{A} is not an ω -model, then $s(\mathfrak{A}) = \omega$. This is because KP is strong enough to prove that $0, 1, 2, 3, \dots \in \mathfrak{A}$, but $\omega \notin \mathfrak{A}$.

But what if \mathfrak{A} is an ω -model?

Notation

Given a tree T and $s \in \omega^{<\omega}$, we define:

$$T/s := \{t \in \omega^{<\omega} : s \frown t \in T\}$$

Theorem

If \mathfrak{A} is an ω -model for KP and $T \in \mathfrak{A}$ is a well-founded tree, then the height function:

$$s \mapsto \|T/s\|$$

is in \mathfrak{A} .

As a consequence, we have that:

Corollary

If \mathfrak{A} is an ω -model of KP_ω , then all $x \in (\omega^\omega)^\mathfrak{A}$, $\omega_1^x \leq s(\mathfrak{A})$. In particular, $\omega_1^{\text{CK}} \leq s(\mathfrak{A})$.

Proof.

Let $x \in (\omega^\omega)^\mathfrak{A}$. For each $\alpha < \omega_1^x$, let T be a well-founded tree, recursive in x , such that $\|T\| \geq \alpha$. Since $x \in \mathfrak{A}$ and recursive functions are Δ_1 functions of arithmetic, it is a Δ_0 function (of set theory), so by Δ_0 -separation we have that $T \in \mathfrak{A}$. By the theorem, the height function $s \mapsto \|T/s\|$ is in \mathfrak{A} , and in particular $\|T\| \subseteq \mathfrak{A}$ (as range of a function is Δ_0). Therefore $\omega_1^x \leq s(\mathfrak{A})$. \square

We now prove the theorem. Recall that since “ T is well-founded” is Π_1^1 , the well-foundedness of T is absolute across all models of sufficiently large fragment of ZF to prove that well-founded trees have a rank function. This would allow us to prove the theorem with an easy induction. Unfortunately, KP is not enough to prove such a statement, so we can’t use that here.

We need a cleverer approach to prove this theorem.

Proof.

We induct on $\|T\|$, and assume that a height function exists for all subtrees T/s , where $s \neq \emptyset$. We define a formula ϕ by:

$$\phi(f, T) \iff f \text{ is a height function for } T$$

ϕ is a Δ_0 formula (Exercise). Applying the induction hypothesis, we have that:

$$\mathfrak{A} \models \forall s \in T \exists f [s = \emptyset \vee \phi(f, T/s)]$$

Applying Δ_0 -collection, we obtain a set $x \in \mathfrak{A}$ such that:

$$\mathfrak{A} \models \forall s \in T \exists f \in x [s = \emptyset \vee \phi(f, T/s)]$$

Proof (Cont.)

Let $a := T \times T \times \bigcup \bigcup \bigcup x$. $a \in \mathfrak{A}$ as KP models are closed under finite Cartesian products (Exercise). Note that if $f \in x$, then $\text{ran}(f) \subseteq \bigcup \bigcup \bigcup x$. By Δ_0 -separation, we may define the function $F(s, t)$ by:

$$F := \{(s, t, \alpha) \in a : \mathfrak{A} \models \exists f \in x[\phi(f, T/s) \wedge f(t) = \alpha]\}$$

Note that for all s, t such that $s \frown t \in T$, we have that:

$$F(s, t) = f(t) = \parallel T/(s \frown t) \parallel$$

where f is the height function for the tree T/s . □

Proof.

Let $b := \text{ran}(F) \cap \mathbf{ORD}$, which is a well-defined set in \mathfrak{A} as range and \mathbf{ORD} are Δ_0 . We see that:

$$\begin{aligned}\sup b &= \sup(\text{ran}(F) \cap \mathbf{ORD}) \\ &= \sup_{s \neq \emptyset} \|T/s\| \\ &= \|T\|\end{aligned}$$

Thus, we let $\alpha_0 := \sup b = \bigcup b$. We may use Δ_0 -separation again to define a function G such that:

$$G(s) := \begin{cases} \alpha_0, & \text{if } s = \emptyset \\ F((k), t), & \text{if } s = (k) \frown t \end{cases}$$

Then G is the height function of T , as desired. □

KP and constructibility

Suppose G is a function on some transitive class A . Recursion theorem asserts that there exists a function F on A such that for all $x \in A$:

$$F(x) = G(F \upharpoonright x)$$

(See Theorem 6.5 of Jech). Using only Δ_1 -separation and Σ_1 -collection, we can get the following theorem of KP:

Theorem (Krivine, Σ_1 -Recursion theorem)

If G is a Σ_1 -function on some transitive class A , then there exists a Σ_1 -function F on A such that for all $x \in A$:

$$F(x) = G(F \upharpoonright x)$$

It turns out that Σ_1 -recursion is sufficient for us to define the function:

$$\alpha \mapsto L_\alpha$$

That is, KP is sufficient to define the constructible hierarchy. This is done with the help of Gödel operations - see more at §12 of Jech.

We get the following result:

Theorem

If \mathfrak{A} is a model of KP_ω , then:

$$\{x \in \mathfrak{A} : \mathfrak{A} \models x \in L\}$$

is a model of $KP + V = L$.

KP is also sufficient to prove the well-known *condensation lemma*:

Theorem

If \mathfrak{A} is a transitive model for $KP + V = L$, then $\mathfrak{A} = L_\alpha$ for some ordinal α .

A model of KP

We've seen that if \mathfrak{A} is an ω -model of KP, then $s(\mathfrak{A}) \geq \omega_1^{\text{CK}}$.

Theorem

There is an ω -model \mathfrak{A} of KP with $s(\mathfrak{A}) = \omega_1^{\text{CK}}$.

We shall borrow a few facts from the theory of hyperarithmetical sets. Define:

$$\text{WFG} := \{\ulcorner T \urcorner : T \text{ is a recursive tree}\}$$

Fact

WFG is a Π_1^1 , but not Σ_1^1 , subset of ω .

Fact

If $x \subseteq \omega$ is not Δ_1^1 , then there exists an ω -model \mathfrak{A} of KP_ω such that $x \notin \mathfrak{A}$.

Lemma

Assume $\omega_1^{\text{CK}} \subseteq \mathfrak{A}$. For all $\alpha < \omega_1^{\text{CK}}$, there exists some $n_\alpha < \omega$ such that if $T \in \mathfrak{A}$ is a recursive tree of height α , then its height function is in $L_{\alpha+n_\alpha}^{\mathfrak{A}}$.

The assumption " $\omega_1^{\text{CK}} \subseteq \mathfrak{A}$ " is to ensure that the set $L_\alpha^{\mathfrak{A}}$ makes sense for all $\alpha < \text{CK}$.

Proof.

We first note that since T is recursive, there exists a formula ϕ , Δ_1 in arithmetic (so Δ_0), such that $T = \{s \in \omega^{<\omega} : \phi(s)\}$, so $T \in L_{\omega+1}$.

We induct on α . We recap the proof that ω -models of KP contains the height function of T . We first observe that there exists some $n'_\alpha < \omega$: such that $\alpha + n'_\alpha = \sup_{\gamma < \alpha} \gamma + n_\gamma$. The induction hypothesis precisely asserts that

$$\mathfrak{A} \models \forall s \in T \exists f \in L_{\alpha+n'_\alpha} [s \neq \emptyset \vee \phi(f, T/s)]$$

where $\phi(f, T/s)$ is the Δ_0 formula asserting that f is the height function of T/s .

Proof (Cont.)

Let $a := T \times T \times \bigcup \bigcup \bigcup L_{\alpha+n'_\alpha}$, and by Δ_0 -separation we may define the function:

$$F := \{(s, t, \xi) \in a : \mathfrak{A} \models \exists f \in L_{\alpha+n'_\alpha} [\phi(f, T/s) \wedge f(t) = \xi]\}$$

We have that $T \in L_{\alpha+n'_\alpha}$ and $\bigcup \bigcup \bigcup L_{\alpha+n'_\alpha} \in L_{\alpha+n'_\alpha+4}$, so $F \in L_{\alpha+n'_\alpha+5}$. We may then define G , the height function of T , from F and a , so $G \in L_{\alpha+n'_\alpha+6}$. Let $n_\alpha := n'_\alpha + 6$, and the induction is complete. □

Proof of Theorem.

Let \mathfrak{A} be an ω -model of KP such that $\text{WFG} \notin \mathfrak{A}$. We wish to show that $\omega_1^{\text{CK}} \notin \mathfrak{A}$. Suppose otherwise. Recall that a tree has a height function iff it is well-founded (in the universe). We may define $L_{\omega_1^{\text{CK}}}^{\mathfrak{A}} \in \mathfrak{A}$, and we see that:

$$x \in \text{WFG}$$

$$\iff$$

$$x \in \omega \wedge \exists f \in L_{\omega_1^{\text{CK}}+4} \exists T \in L_{\omega+1}[x = \ulcorner T \urcorner \wedge \varphi(f, T)]$$

The formula on RHS is Δ_0 , so $\text{WFG} \in \mathfrak{A}$, a contradiction. \square

Well-founded models of KP

Here are some heads up of what we will be covering after summer school.

We say that an element $x \in A$ is *well-founded* if there does not exist an E -decreasing sequence below x . Consider the following way to extract the well-founded part of \mathfrak{A} :

$$B := \{x \in A : x \text{ is well-founded}\}$$

and consider the structure $\mathfrak{B} := (B, E)$. By construction, \mathfrak{B} is a well-founded structure.

Theorem

\mathfrak{B} is a well-founded ω -model of KP.

Corollary

$L_{\omega_1^{\text{CK}}}$ is a well-founded ω -model of KP.

I plan to prove these two results after summer school.

Admissible ordinals

Models of KP of the form L_α are of much interest, and are very important in the proof of $\Sigma_1^1\text{-AD} \rightarrow 0^\sharp$ exists.

Definition

A countable ordinal α is *admissible* if $L_\alpha \models \text{KP}$.

Theorem

α is *admissible* iff $\alpha = \omega_1^x$ for some real x .