NUS Reading Seminar Summer 2023

Session 6

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Kripke-Platek Set Theory

Recall that the Kripke-Platek KP set theory consists of the axioms of extensionality, pairing, union, regularity, Δ_0-separation and Δ_0-collection.

Lemma

Both Δ_1-separation and Σ_1-collection hold in KP*.*

Ordinals

Let $\mathfrak{A}$ be a model of KP. $\textbf{ORD}^{\mathfrak{A}}$ may not be a well-order (from the universe's perspective), but it has an infinite initial segment - i.e. $\omega^{\mathfrak{A}}$. Note that $\omega^{\mathfrak{A}}$ need not be an element of $\mathfrak{A}$!

Notation

Let $\mathfrak{A} \models \mathsf{KP}$. Then:

$$s(\mathfrak{A}) := \sup\{\mathsf{otp}(S) : S \text{ is an initial segment of } \textbf{ORD}^{\mathfrak{A}} \\ \wedge S \text{ is well-ordered}\}$$

We have $s(\mathfrak{A}) > \omega$ iff $\omega^{\mathfrak{A}} \in \mathfrak{A}$ iff $\mathfrak{A}$ is a standard model. Note that a standard model need not be well-founded.

Models of KP

We previously "proved" the following theorem:

Theorem

There is an ω-model $\mathfrak{A}$ of KP with $s(\mathfrak{A}) = \omega_1^{\mathsf{CK}}$.

$\mathfrak{A}$ may be standard, but how do we get well-founded models?

Let $\mathfrak{A} = (A, E)$ be a model of KP. Recall that models of KP satisfies the following Σ_1-Recursion theorem:

Theorem (Krivine, Σ_1-Recursion theorem)

If G is a Σ_1-function on some transitive class A, then there exists a Σ_1-function F on A such that for all $x \in A$:

$$F(x) = G(F \restriction x)$$

We can then (internally) define a Σ_1 rank function in the model.

We say that an element $x \in A$ is *well-founded* if there does not exist an E-decreasing sequence below x. Consider the following way to extract the well-founded part of $\mathfrak{A}$:

$$B := \{x \in A : x \text{ is well-founded}\}$$

and consider the structure $\mathfrak{B} := (B, E)$. Note that if x is a well-founded, then $\rho(x) < \sigma$ for all non-standard ordinals σ.

Theorem

$\mathfrak{B} = (B, E)$ *is a well-founded* ω*-model of* KP.

Proof.

Extensionality, pairing, union, and Δ_0-separation are easy to check (as all sets involved are well-founded). For regularity, note that if $x \in B$ is such that $\phi(x)$, then there's an E-minimum element in $\mathrm{trcl}(\{x\})$ that satisfies $\phi(x)$. Then that element witnesses regularity for ϕ. It remains to show that Δ_0-collection holds. We shall assume that $\mathfrak{A} \neq \mathfrak{B}$, for otherwise the result follows immediately.

Proof.

Let ϕ be Δ_0. Fix any $x \in B$, and suppose $\mathfrak{B} \models \forall y \in x \, \exists z \, \phi(y, z)$. Let $\rho \in A$ be the rank function for $\mathfrak{A}$. Fix any $y \in x$, and let $z \in B$ such that $\mathfrak{B} \models \phi(y, z)$. By the absoluteness of Δ_0-formulas, $\mathfrak{A} \models \phi(x, y)$. Since y is well-founded, $\rho(y) < \sigma$ for some fixed non-standard ordinal $\sigma \in A$. Thus:

$$\mathfrak{A} \models \forall y \in x \, \exists z [\phi(y, z) \wedge \rho(z) < \sigma)]$$

Note that $\rho(z) < \sigma$ is Δ_1, as ρ is a Σ_1 function and so both $\rho(z) < \sigma$ and $\rho(z) \geq \sigma$ are Σ_1. By Δ_1-collection in $\mathfrak{A}$, we have some $u' \in A$ such that:

$$\mathfrak{A} \models \forall y \in x \, \exists z \in u' [\phi(y, z) \wedge \rho(z) < \sigma)]$$

Proof (Cont.)

The entire formula above is Δ_1, so by Δ_1-separation of $\mathfrak{A}$, we have that:

$$X := \{\tau < \omega : \mathfrak{A} \models \forall y \in x\, \exists z \in u'[\phi(y,z) \wedge \rho(z) < \tau)]\}$$

is a well-defined element of A. X contains all non-standard ordinals of $\mathfrak{A}$, so $\sigma_0 := \inf X$ is a standard ordinal (for otherwise σ_0 is the least non-standard ordinal).

Proof (Cont.)

Now let:

$$u := \{w \in u' : \mathfrak{A} \models \rho(w) \leq \sigma_0\}$$

u is a well-defined element of A by Δ_1-separation. Since $\rho(u) \leq \sigma_0 + 1$, $u \in B$. We see that u witnesses Δ_0-replacement in $\mathfrak{B}$ for ϕ: For any $y \in x$, there exists some well-founded $z \in u$ such that $\phi(y, z)$ holds as $\rho(z)$ is below every non-standard ordinal. Note that if $\omega^{\mathfrak{A}}$ exists, then it is well-founded so $\omega^{\mathfrak{B}} = \omega^{\mathfrak{A}}$ exists. □

Corollary

$L_{\omega_1^{\mathrm{CK}}}$ is an ω-model for KP.

Proof.

Let $\mathfrak{A}$ be an ω-model of KP with $s(\mathfrak{A}) = \omega_1^{\mathrm{CK}}$. Then $L^{\mathfrak{A}}$ is an ω-model of $\mathrm{KP} + \mathrm{V} = \mathrm{L}$. But by the absoluteness of constructibility and Gödel's condensation lemma, we must have $L^{\mathfrak{A}} = L_{\omega_1^{\mathrm{CK}}}$. □

Admissible ordinals

Definition

Let $x \in \omega^\omega$. A countable ordinal α is *admissible in x* if $L_\alpha[x] \models \mathsf{KP}$. α is *admissible* if it is admissible in $(0, 0, 0, \dots)$.

We just proved that ω_1^{CK} is an admissible ordinal. Since if $L_\alpha \models \mathsf{KP}$, $s(L_\alpha) = \alpha \geq \omega_1^{\mathsf{CK}}$, we have that ω_1^{CK} is the least admissible ordinal.

Theorem (Sacks)

A countable ordinal α is admissible in x iff $\alpha = \omega_1^y$ for some $x \leq_{\mathrm{T}} y$.

The proof uses a tool called $\epsilon\sigma$-logic, which I shall not discuss.

Recall the following theorem:

Theorem

The following are equivalent:

1. *Analytic determinacy (Σ^1_1-AD).*
2. $0^\sharp$ *exists.*
3. *There exists an non-trivial elementary embedding* $j : L \to L$.

We proved (2) $\implies$ (1), and (2) $\iff$ (3) is a deep theorem in set theory. We will begin setting up the proof of (1) $\implies$ (3).

Friedman's set

The *Friedman's set* is a set $\mathrm{F} \subseteq \omega^\omega$ with some special properties. We will use F to prove Σ^1_1-AD implies $0^\sharp$ as follows:

1. F is Σ^1_1.
2. F intersects every Turing cone.
3. By Σ^1_1-AD, every Σ^1_1 set either contains a cone or is disjoint from a cone. Thus, F contains a cone C.
4. If $x \in C \subseteq \mathrm{F}$, then ω_1^x is a cardinal in L.
5. If every admissible ordinal is a cardinal in L, then $j : L \to L$ exists.

Definition

The *Friedman's set* is the set defined by:

$$F := \{x \in \omega^\omega : \forall \alpha < \omega_1^x \, \forall y \subseteq \alpha [y \in L_{\omega_1^x} \to y \in L_{\alpha+3}[x]]\}$$

Heuristically, Friedman's set is the set of reals which speeds up constructibility.

Lemma

F *is* Σ^1_1.

The proof is a technical (but not difficult) analysis of hyperarithmetic set theory, so we shall take this fact for granted.

Lemma

F *intersects every Turing cone. That is, for all* x *there exists some* $y \in \mathsf{F}$ *such that* $x \leq_{\mathrm{T}} y$*.*

Proof.

Fix any $x \in \omega^\omega$. Let $\mathfrak{A}$ be an ω-model $\mathfrak{A} = (A, E)$ of KP + "All sets are countable", with $x \in \mathfrak{A}$. By taking the ultrapower using a non-principal ultrafilter over $\omega^{\mathfrak{A}}$, we may assume that $\mathfrak{A}$ is not well-founded. Note that x is in the ultrapower (or at least there is a real Turing equivalent to x).

Let $\sigma \in \mathbf{ORD}^{\mathfrak{A}}$ be non-standard. Since $L_\sigma^{\mathfrak{A}} \in \mathfrak{A}$, $\mathfrak{A} \models L_\sigma$ is countable, so there exists some real $y \in A$ that codes a relation $R \in A$ on $\omega^{\mathfrak{A}}$, such that $(\omega, R) \cong (L_\sigma^{\mathfrak{A}}, E)$. We shall show that $z := x \oplus y \in \mathrm{F}$, the Friedman's set.

Proof (Cont.)

Let $\alpha < \omega_1^z$, and let $w \subseteq \alpha$. Since $z \in A$, all trees recursive in z are in A, so $s(\mathfrak{A}) \geq \omega_1^z$. This implies that $\omega_1^z \leq \sigma$, so:

1. α is a standard ordinal in $\mathfrak{A}$, so $\alpha < \sigma$. In particular, $\mathfrak{A} \models \alpha \in L_\sigma$.
2. $\mathfrak{A} \models w \in L_{\omega_1^z} \subseteq L_\sigma$.

Using the relation R, we obtain two integers n_α, n_w in $\mathfrak{A}$ such that:

1. $x_\alpha := \{n : n \, R \, n_\alpha\}$ is isomorphic to (α, E).
2. $x_w := \{n : n \, R \, n_w\}$ is isomorphic to (w, E).

Proof (Cont.)

Let $i : x_\alpha \to \alpha$ be the order isomorphism. We see that $i \in L_{\alpha+3}[z]$: x_α is a subset of $\omega^{\mathfrak{A}}$ which is definable from α and y, so $x_\alpha \in L_{\alpha+1}[z]$. Then i can be constructed in the level $L_{\alpha+3}[z]$. Since $x_w \in L_{\omega+1}[y] \subseteq L_{\omega+1}[z]$, we have that $w = i[x_w] \in L_{\alpha+3}[z]$. □

By the same argument as Martin's cone theorem, F contains a cone.

Let $x \in \omega^\omega$ such that $C_x \subseteq \mathrm{F}$. Then for all $z \in C_x \subseteq \mathrm{F}$ implies that:

$$\forall \alpha < \omega_1^z \, \forall y \subseteq \alpha [y \in L_{\omega_1^z} \to y \in L_{\alpha+3}[z]]$$

Lemma

For any $z \in C_x \subseteq \mathrm{F}$:

$$\forall \alpha < \omega_1^z \, \forall y \subseteq \alpha [y \in L \to y \in L_{\omega_1^z}[z]]$$

We will take this lemma for granted.

Lemma

If $C \subseteq \mathsf{F}$ is a Turing cone and $x \in C$, then ω_1^x is a cardinal in L.

Proof.

Fix $x \in C \subseteq \mathsf{F}$, and suppose ω_1^x is not a cardinal in L. Let $\alpha < \omega_1^x$ such that:

$$L \models |\alpha| = |\omega_1^x|$$

Recall that ZFC proves that this implies that there exists a well-order $\preceq$ on α such that $(\alpha, \preceq)$ has order-type ω_1^α. Let $\Gamma : \mathbf{ORD} \times \mathbf{ORD}$ be the canonical pairing function. Then $\Gamma[\preceq] \subseteq \alpha^2 < \omega_1^x$ (note that if there exists a tree recursive in x of height α, then one may construct a tree recursive in x of height α^2).

Proof (Cont.)

By the previous lemma, $\preceq \in L_{\omega_1^x}[x]$. We recall the following theorem we proved before summer school:

Theorem. If $\mathfrak{A} = (A, E)$ is an ω-model of KP and $T \in \mathfrak{A}$ is a well-founded tree, then the height function $s \mapsto \|T/s\|$ is in A. In particular, $T \in A$.

Since α^2 is countable, $\preceq$ defines a countable, well-founded tree of height ω_1^x (by using the bijection between α^2 and ω). Since ω_1^x is admissible in x by Sack's theorem, $L_{\omega_1^x}[x] \models \text{KP}$, so applying the theorem we have that $\omega_1^x \in L_{\omega_1^x}[x]$, a contradiction. $\square$

Coupled with Sack's theorem, the lemma can be rephrased as follows: Suppose $C_x \subseteq \mathrm{F}$ with apex $x \in \omega^\omega$. If α is a countable ordinal admissible in x, then α is a cardinal in L.

Lemma

In fact, every ordinal admissible in x, countable or not, is a cardinal in L.

We'll prove this next week.