NUS Reading Seminar Summer 2023

Session 7

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Friedman Set

Let F denote the Friedman's set, and let $C \subseteq$ F be a cone.

Lemma

If $x \in C \subseteq$ F and α is admissible in x, then α is a cardinal in L.

Last week, we proved this in the case where α is countable. Note that every uncountable regular cardinal is admissible in x for any $x \in \omega^\omega$.

Proof.

Let α be an uncountable ordinal admissible in x. Since $x \in C \subseteq \mathsf{F}$, $C_x \subseteq \mathsf{F}$. Observe that:

$$C_x \subseteq \mathsf{F} \iff \forall y \in \omega^\omega [x \leq_{\mathrm{T}} y \to y \in \mathsf{F}]$$

Since F is Σ^1_1, the above statement is $\Pi^1_2(x)$. By Shoenfield's absoluteness, "$C_x \subseteq \mathsf{F}$" is absolute across all models of ZF containing x.

Proof (Cont.)

Let $\mathbb{P}$ be any forcing notion that collapses α to a countable ordinal (for example, $\mathbb{P} = \mathrm{Coll}(\omega, |\alpha|^+)$). Let G be $\mathbb{P}$-generic over V. Then $\alpha \in V[G]$ is a countable ordinal. By the absoluteness of constructibility, $L_\alpha^{V[G]}[x] = L_\alpha^V[x]$ is a model of KP. Therefore, α is admissible in x, so by the theorem in the case of countable ordinals, α is a cardinal in $L^{V[G]}$. But $L^{V[G]} = L^V$, again by the absoluteness of constructibility. □

Main Theorem

We are now ready to prove the main theorem.

Theorem

If for some $x \in \omega^\omega$, every ordinal admissible in x is a cardinal of L, then there is a non-trivial elementary embedding $j : L \to L$.

A common way to construct an elementary embedding is to use ultrapowers. Let $(M, \in)$ be a model. Recall that if $U \in M$ is an ultrafilter on some cardinal κ, then we may define the equivalence relations by:

$$f \sim g \iff \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in U$$

Let M^κ/U be the set of equivalence relations $[f]$. Then:

$$[f] \in^* [g] \iff \{\alpha < \kappa : f(\alpha) \in g(\alpha)\} \in U$$

Note that elementary embeddings of ultrapowers are always non-trivial.

It turns out that we do not actually need the fact that $U \in M$.

Definition

Let $(M, \in)$ be a model. U (not necessarily in M) is an *M-ultrafilter* on κ if:

1. $\emptyset \notin U$, $\kappa \in U$.
2. $X \in U$ and $Y \in U$ implies that $X \cap Y \in U$.
3. If $X \in U$, $X \subseteq Y$ and $Y \in M$. then $Y \in U$.
4. If $X \subseteq \kappa$ and $X \in M$, then $X \in U$ or $Y \in U$.

We can proceed with the usual construction of M^κ/U.

Theorem

If U is a non-principal M-ultrafilter, then $j : M \rightarrow M^\kappa/U$ is a non-trivial elementary embedding.

If U is closed under all countable intersections (including those not in M), then M^κ/U is well-founded. See Lemma 17.2 of Jech.

We now proceed with the proof. Since L is absolute, it suffices to prove this theorem in $L[x]$. The steps to proving this theorem are as follows:

1. Construct an elementary submodel $(M, \in) \preceq (L_{\aleph_3}[x], \in)$ with special properties.
2. Let π be the inverse of the transitive collapse of $(M, \in)$. Then $\pi : L_\alpha[x] \to M$ is elementary and non-trivial.
3. Use π to define a non-principal L-ultrafilter D on some κ that is closed under all countable intersections.

Then L^κ/D is well-founded, so its transitive collapse is a subclass of L that is elementarily equivalent to L. By Gödel's condensation lemma, its transitive collapse is exactly L.

The model M we wish to construct is an elementary submodel $M \preceq L_{\aleph_3}[x]$ such that:

1. $|M| = \aleph_1$.
2. $\aleph_2 \in M$.
3. M is closed under countable sequences (i.e. $M^\omega \subseteq M$).

We shall do this by induction.

1. By Löwenheim-Skolem, we let $M_0 \preceq L_{\aleph_3}[x]$ be any elementary submodel such that $|M_0| = \aleph_1$ and $\aleph_2 \in M_0$.
2. By Löwenheim-Skolem again, we let $M_{\alpha+1} \preceq L_{\aleph_3}[x]$ be any elementary submodel such that $|M_{\alpha+1}| = \aleph_1$ and $M_\alpha \cup M_\alpha^\omega \subseteq M_{\alpha+1}$. Note that $|M_\alpha^\omega| = \aleph_1^{\aleph_0} = \aleph_1$, as $L[x] \models \mathrm{CH}$ (Exercise).
3. If α is limit, let $M_\alpha := \bigcup_{\beta<\alpha} M_\beta$. $M_\alpha \preceq L_{\aleph_3}[x]$ as the increasing union of elementary submodel is an elementary submodel (Exercise).

We then let $M := \bigcup_{\alpha<\omega_1} M_\alpha$. Clearly M satisfies the three required conditions, so this completes step 1. Note that $M \models \mathrm{KP}$, as $\aleph_3$ is admissible in x.

Let $\pi : L_\alpha[x]$ be the inverse of transitive collapse of M. Note that $|\alpha| = \aleph_1$. We then define:

$$D := \{Z \subseteq \kappa : \kappa \in \pi(Z)\}$$

Claim

D is a non-principal L-ultrafilter over κ that is closed under all countable intersections.

Note that D is clearly a filter.

Proof.

We first note that since $L_\alpha[x] \preceq L_{\aleph_3}[x]$, and $L_{\aleph_3}[x] \models \text{KP}$, we have that α is admissible in x. By our hypothesis, α is a cardinal in L. Therefore, since $\alpha > \kappa$, $Z \in L_\alpha \subseteq L_\alpha[x]$ for all $Z \subseteq \kappa$.

D is an L-ultrafilter: We have that for all $Z \subseteq \kappa$, $Z \in \text{dom}(\pi) = L_\alpha[x]$, so:

$$\kappa \in \pi(\kappa) = \pi(Z \cup (\kappa \setminus Z)) = \pi(Z) \cup \pi(\kappa \setminus Z)$$

Proof (Cont.)

D is countably closed: Suppose $\{Z_n : n < \omega\} \subseteq D \subseteq L_\alpha[x]$. Then $\{\pi(Z_n) : n < \omega\} \subseteq M$, and since M is closed under countable sequences. $\{\pi(Z_n) : n < \omega\} \in M$. Since $M \models \mathrm{KP}$ and intersection is Δ_0, we have that:

$$\bigcap_{n<\omega} \pi(Z_n) = \bigcap \{\pi(Z_n) : n < \omega\} \in M$$

Let $Z := \pi^{-1}\left(\bigcap_{n<\omega} Z_n\right) \in L_\alpha[x]$. By elementarity, $Z = \bigcap_{n<\omega} Z_n$. We have that:

$$\kappa \in \bigcap_{n<\omega} \pi(Z_n) = \pi\left(\bigcap_{n<\omega} Z_n\right) = \pi(Z)$$

so $Z \in D$. □

Some remarks on higher determinacy:

1. Σ^1_n-AD is equiconsistent with $\mathsf{ZFC} + n$ Woodin cardinals. Consequently, projective determinacy (i.e. Σ^1_n-AD for all n) is equiconsistent with the conjunction of $\mathsf{ZFC} + n$ Woodin cardinals for all n.
2. Recall that AD is equiconsistent with $\mathsf{ZFC} + \omega$ Woodin cardinals. It turns out that this is strictly stronger than $\mathsf{ZFC} + n$ Woodin cardinals for all n.
3. *Turing determinacy* asserts that every Turing invariant subset of ω^ω is determined. It is open if it is equiconsistent with AD.

$0^\sharp$ and Elementary embeddings

In this section, we discuss the proof that a non-trivial $j : L \rightarrow L$ exists iff Silver indiscernibles exist.

We first introduce Skolem terms and functions.

Definition

Fix a model $\mathfrak{A} = (A, \in)$, and let $\varphi(u, v_1, \ldots, v_n)$ be a formula. A *Skolem function* for φ is an n-ary function $h^{\mathfrak{A}}_{\varphi}$ such that for all $a_1, \ldots, a_n \in A$, if:

$$\exists a \in A[\mathfrak{A} \models \varphi[a, a_1, \ldots, a_n]]$$

then:

$$\mathfrak{A} \models \varphi[h^{\mathfrak{A}}_{\varphi}(a_1, \ldots, a_n), a_1, \ldots, a_n]$$

There exists a Skolem function in L for all formulas φ, as L is globally well-ordered by $<_L$, so we may let:

$$h^L_\varphi(v_1, \dots, v_n) := \begin{cases} \text{The } <_L\text{-least } u \text{ such that } \varphi^L(u, v_1, \dots, v_n) \\ \emptyset, \text{otherwise} \end{cases}$$

Definition

A *Skolem term* $t(v_1, \ldots, v_n)$ is a term made by the composition of Skolem functions and variables.

If $0^\sharp$ exists, then every element can be expressed as $t^L[\gamma_1, \ldots, \gamma_n]$ for some indiscernibles $\gamma_1 < \cdots < \gamma_n$.

Proof of $0^\sharp \implies j : L \to L$.

Let $j : I \to I$ be any order-preserving function from the indiscernibles into itself. Extend j to L by:

$$j(t^L[\gamma_1, \ldots, \gamma_n]) := t^L[j(\gamma_1), \ldots, j(\gamma_n)]$$

Clearly j is elementary, and if $j : I \to I$ is non-trivial, then so is $j : L \to L$. □

We now give a sketch of the proof of the converse. Let γ be the critical point of $j : L \to L$, and assume WLOG that it is the ultrapower embedding $j : L \to L^\gamma/D$, where $D = \{X \in \mathcal{P}^L(\gamma) : \gamma \in j(X)\}$.

A combinatorial argument gives:

Lemma

If κ is a limit cardinal and $\mathrm{cf}(\kappa) > \gamma$, then $j(\kappa) = \kappa$.

Now define a sequence of classes as follows:

1. $U_0 := \{\kappa \in \mathbf{ORD} : \kappa \text{ is a limit cardinal} \wedge \mathrm{cf}(\kappa) > \gamma\}$.
2. $U_{\alpha+1} := \{\kappa \in U_\alpha : |U_\alpha \cap \kappa| = \kappa\}$.
3. $U_\alpha := \bigcap_{\beta<\alpha} U_\beta$, if α is limit.

One can check that U_α is a proper class for all α.

Definition

Let $\mathfrak{A} = (A, \in)$ be a model. Given a set $X \subseteq A$, the *Skolem hull* of X in $\mathfrak{A}$ is:

$$H^{\mathfrak{A}}(X) := \{t^{\mathfrak{A}}[x_1, \ldots, x_n] : x_1, \ldots, x_n \in X\}$$

Some basic properties of Skolem hull:

1. $H^{\mathfrak{A}}(X) \preceq \mathfrak{A}$.
2. If X is infinite, then $|H^{\mathfrak{A}}(X)| = |X|$.
3. If $X \subseteq Y$, then $H^{\mathfrak{A}}(X) \subseteq H^{\mathfrak{A}}(Y)$.

Fix $\kappa \in U_{\omega_1}$, so $|U_\alpha \cap \kappa| = \kappa$ for all $\alpha < \omega_1$. For $\alpha < \omega_1$, define:

$$M_\alpha := H^{L_\kappa}(\gamma \cup (U_\alpha \cap \kappa))$$

We have $M_\alpha \preceq L_\kappa$. Let $\pi_\alpha : M_\alpha \to L_\kappa$ be the transitive collapse (as $|M_\alpha| = \kappa$). Then $i_\alpha := \pi_\alpha^{-1} : L_\kappa \to L_\kappa$ is an elementary embedding. Let $\gamma_\alpha := i_\alpha(\gamma)$.

Lemma

1. *γ_α is the least ordinal greater than γ in M_α.*
2. *If $\alpha < \beta$ and $x \in M_\beta$, then $i_\alpha(x) = x$. In particular, $i_\alpha(\gamma_\beta) = \gamma_\beta$.*
3. *If $\alpha < \beta$, then $\gamma_\alpha < \gamma_\beta$.*

Proof.

We first observe that $j \restriction M_\alpha$ is the identity. Indeed, if $x = t^L[\eta_1, \ldots, \eta_n] \in M_\alpha = H^{L_\kappa}(\gamma \cup (U_\alpha \cap \kappa))$, then:

$$j(x) = j(t[\eta_1, \ldots, \eta_n]) = t[j(\eta_1), \ldots, j(\eta_n)] = t[\eta_1, \ldots, \eta_n] = x$$

as j fixes all elements in $\gamma \cup (U_\alpha \cap \kappa)$.

Proof (Cont.)

1. Elementarity asserts that $i_\alpha(\gamma)$ is the least ordinal in M_α that is $\geq \gamma$. But $\gamma \notin M_\alpha$, as $j(\gamma) \neq \gamma$.
2. Write $x = t^L[\eta_1, \ldots, \eta_n]$, where $\eta_1, \ldots, \eta_n \in \gamma \cup (U_\beta \cap \kappa)$. If $\eta \in \gamma$, clearly $i_\alpha(\eta) = \eta$ as $\gamma \subseteq M_\alpha$. If $\eta \in U_\beta \cap \kappa$, then since $U_\beta \subseteq U_\alpha$, $|\eta \cap (U_\alpha \cap \kappa)| = \kappa$, so $\pi_\alpha(\eta) = \eta$. Hence $i_\alpha(x) = x$.
3. Since $M_\beta \subseteq M_\alpha$, $\gamma_\alpha \leq \gamma_\beta$. $\gamma_\alpha \neq \gamma_\beta$ as $i_\alpha(\gamma_\alpha) > i_\alpha(\gamma) = \gamma_\alpha$, but $i_\alpha(\gamma_\beta) = \gamma_\beta$.

□

Lemma

If $\alpha < \beta < \omega_1$, then there exists an elementary embedding $i_{\alpha,\beta} : L_\kappa \to L_\kappa$ such that:

1. *If $\xi < \alpha$ or $\xi > \beta$, then $i_{\alpha,\beta}(\gamma_\xi) = \gamma_\xi$.*
2. *$i_{\alpha,\beta}(\gamma_\alpha) = \gamma_\beta$.*

Proof.

Let $M_{\alpha,\beta} = H^{L_\kappa}(\gamma_\alpha \cup (U_\beta \cap \kappa)$, and let $i_{\alpha,\beta} = \pi^{-1}_{\alpha,\beta}$ be the inverse of the transitive collapse. Property (1) of $i_{\alpha,\beta}$ is proved the same way as (2) of the previous lemma. For property (2) of $i_{\alpha,\beta}$, we first observe that:

1. Since $\gamma_\alpha \subseteq M_{\alpha,\beta}$, $i_{\alpha,\beta}(\gamma_\alpha)$ is the least ordinal at least γ_α in $M_{\alpha,\beta}$.
2. Since $\gamma_\beta \in M_{\alpha,\beta}$ and $\gamma_\alpha < \gamma_\beta$, $i_{\alpha,\beta}(\gamma_\alpha) \leq \gamma_\beta$.

It suffices to show that there are no ordinals $\delta \in M_{\alpha,\beta}$ such that $\gamma_\alpha \leq \delta < \gamma_\beta$.

Proof (Cont.)

Suppose such a δ exist. Write $\delta = t(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_k)$, where $\xi_i < \gamma_\alpha$ and $\eta_i \in U_\beta \cap \kappa$. Then:

$$(L_\kappa, \in) \models \exists \xi_1, \ldots, \xi_n < \gamma_\alpha [\gamma_\alpha \leq t[\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_k] < \gamma_\beta]$$

By the previous lemma, we may rewrite above as:

$$(L_\kappa, \in) \models \exists \xi_1, \ldots, \xi_n < i_\alpha(\gamma) \\ [i_\alpha(\gamma) \leq t[\xi_1, \ldots, \xi_n, i_\alpha(\eta_1), \ldots, i_\alpha(\eta_k)] < i_\alpha(\gamma_\beta)]$$

By elementarity:

$$(L_\kappa, \in) \models \exists \xi_1, \ldots, \xi_n < \gamma [\gamma \leq t[\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_k] < \gamma_\beta]$$

But this contradicts the minimality of γ_β in M_β (as $t[\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_k] \in M_\beta$). □

We now prove the converse. We shall borrow the following result:

Theorem

$0^\sharp$ *exists iff for some limit ordinal* λ, $(L_\lambda, \in)$ *has an uncountable set of indiscernibles.*

See Corollary 18.18 of Jech.

Proof of $j : L \to L \implies 0^\sharp$.

We shall show that $\{\gamma_\alpha : \alpha < \omega_1\}$ is a set of indiscernibles for $(L_\kappa, \in)$. Given a formula φ, we wish to show that for all $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$, we have that:

$$L_\kappa \models \varphi[\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_n}] \iff L_\kappa \models \varphi[\gamma_{\beta_1}, \ldots, \gamma_{\beta_n}]$$

Let $\delta_1 < \cdots < \delta_n$ such that $\alpha_n, \beta_n < \delta_1$. Applying the embedding i_{α_n,δ_n}, we have that:

$$L_\kappa \models \varphi[\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_n}] \iff L_\kappa \models \varphi[\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_{n-1}}, \gamma_{\delta_n}]$$

as i_{α_n,δ_n} fixes $\alpha_1, \ldots, \alpha_{n-1}$. Repeat this for $i_{\alpha_{n-1},\delta_{n-1}}, i_{\alpha_{n-2},\delta_{n-2}}, \ldots$, and we get:

$$L_\kappa \models \varphi[\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_n}] \iff L_\kappa \models \varphi[\gamma_{\delta_1}, \ldots, \gamma_{\delta_n}]$$

Now do the same for $\varphi[\gamma_{\beta_1}, \ldots, \gamma_{\beta_n}]$. $\square$