Weak A2 spaces, the Kastanas game and strategically Ramsey sets

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Infinite-dimensional Ramsey theory

Infinite-dimensional Ramsey theory started with the study of infinite subsets of natural numbers, $[\mathbb{N}]^\infty$. Let's recall the definition.

Notation

Given $A \in [\mathbb{N}]^\infty$ and $a \in [A]^{<\infty}$, we write:

$$[a, A] := \{B \in [A]^\infty : a \sqsubseteq B\}$$

where $a \sqsubseteq B$ means that $B \cap \max(a) = a$.

Definition

A subset $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is *Ramsey* if for all $A \in [\mathbb{N}]^\infty$ and $a \in [A]^{<\infty}$, there exists some $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \subseteq \mathcal{X}^c$.

Definition (Kastanas)

Let $A \in [\mathbb{N}]^\infty$, and let $a \in [A]^{<\infty}$. The *Kastanas game* played below $[a, A]$, denoted as $K[a, A]$, is:

$$
\begin{array}{c|ccccccc}
\textbf{I} & A_0 \subseteq A & & A_1 \subseteq B_0 & & A_2 \subseteq B_1 & \cdots \\
\hline
\textbf{II} & & x_0 \in A_0 & & x_1 \in A_1 & & \cdots \\
 & & B_0 \subseteq A_0 & & B_1 \subseteq A_1 & & \cdots
\end{array}
$$

where:

- $\max(a) < x_0 < x_1 < \cdots$.
- A_n, B_n are infinite subsets of $\mathbb{N}$.

The outcome of the game is $a \cup \{x_0, x_1, \ldots\}$.

Definition

We say that **I** (similarly **II**) has a strategy in $K[a, A]$ to *reach* $\mathcal{X} \subseteq \mathcal{R}$ if it has a strategy in $K[a, A]$ to ensure that the outcome is in $\mathcal{X}$. We write it as:

$$\mathbf{I} \xrightarrow{K[a,A]} \mathcal{X}$$

Definition

A set $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is *Kastanas Ramsey* if for all $A \in [\mathbb{N}]^\infty$ and $a \in [A]^{<\infty}$, there exists some $B \in [a, A]$ such that one of the following holds:

1. **I** has a strategy in $K[a, B]$ to reach $\mathcal{X}^c$.
2. **II** has a strategy in $K[a, B]$ to reach $\mathcal{X}$.

Observe that if $\mathcal{X}$ is Ramsey, then $\mathcal{X}$ is Kastanas Ramsey. Let $A \in [\mathbb{N}]^\infty$ and $a \in [A]^{<\infty}$.

1. If $[a, B] \subseteq \mathcal{X}^c$ for some $B \in [a, A]$, then **I** can always reach $\mathcal{X}^c$ in $K[a, B]$.
2. If $[a, B] \subseteq \mathcal{X}$ for some $B \in [a, A]$, then **II** can always reach $\mathcal{X}$ in $K[a, B]$.

Theorem (Kastanas)

A set $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is Ramsey iff it is Kastanas Ramsey.

The Kastanas game can be easily generalised to other settings. One such example is vector spaces. Let E be a vector space over a field $\mathbb{F}$.

1. Start with an infinite-dimensional subspace $A \subseteq E$, and a finite-dimensional subspace $a \subseteq A$.
2. A_n, B_n's will be infinite-dimensional subspaces instead of subsets.
3. $x_0, x_1, \ldots$ will be vectors, and to make sure that we can an infinite-dimensional subspace at the end, we ask thaht:

$$x_n \notin \operatorname{span}(a \cup \{x_0, \ldots, x_{n-1}\})$$

We can then define Kastanas Ramsey subsets accordingly.

There are also many other settings/examples where we may define the Kastanas game.

Goal. Find an abstract framework which captures all such examples, and then define an abstract version of Kastanas game.

Todorčević's axioms

Consider the following setting:

1. $\mathcal{R}$ is a non-empty set, representing some set of finite increasing sequences.
   - For $\mathcal{R} = [\mathbb{N}]^\infty$, every $A \in [\mathbb{N}]^\infty$ is an increasing sequence of natural numbers.
2. $\leq$ is a quasi-order on $\mathcal{R}$.
   - For $\mathcal{R} = [\mathbb{N}]^\infty$, consider the partial order $\subseteq$.
3. $\mathcal{AR}$ is a non-empty set, representing the set of finite increasing sequences.
   - For $\mathcal{R} = [\mathbb{N}]^\infty$, we have $\mathcal{AR} = [\mathbb{N}]^{<\infty}$.
4. $r : \mathcal{R} \times \omega \to \mathcal{AR}$ is a function, with $r_n(-) := r(-, n)$, is a restriction map that takes the first n elements of the sequence.
   - If $A = \{x_0, x_1, \dots\} \in [\mathbb{N}]^\infty$, then $r_n(A) = \{x_0, \dots, x_{n-1}\}$.

Notation

Given a triple $(\mathcal{R}, \leq, r)$, for $A \in \mathcal{R}$ and $a \in \mathcal{AR}$:

$$a \sqsubseteq A \iff a = r_n(A) \text{ for some } n \in \mathbb{N}$$

Notation

If $A \in \mathcal{R}$ and $a \in \mathcal{AR}$,

$$[a, A] := \{B \in \mathcal{AR} : a \sqsubseteq B \wedge B \leq A\}$$

Axiom (**A1**, Sequencing)

(1) $r_0(A) = \emptyset$ for all $A \in \mathcal{AR}$.

(2) $A \neq B$ implies $r_n(A) \neq r_n(B)$ for some n.

(3) $r_n(A) = r_m(B)$ implies $n = m$ and $r_k(A) = r_k(B)$ for all $k < n$.

Axiom (**A2**, Finitisation)

There is a quasi-ordering $\leq_{\text{fin}}$ on $\mathcal{AR}$ such that:

(1) $\{a \in \mathcal{AR} : a \leq_{\text{fin}} b\}$ is finite for all $b \in \mathcal{AR}$.

(2) $A \leq B$ iff $\forall n \exists m [r_n(A) \leq_{\text{fin}} r_m(B)]$.

(3) $\forall a, b \in \mathcal{AR}[a \sqsubseteq b \land b \leq_{\text{fin}} c \to \exists d \sqsubseteq c[a \leq_{\text{fin}} d]]$.

Axiom (A1, Sequencing)

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Axiom (A2, Finitisation)

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(2) $A \leq B$ iff $\forall n \exists m [r_n(A) \leq_{\mathrm{fin}} r_m(B)]$.
(3) $\forall a, b \in \mathcal{AR} [a \sqsubseteq b \land b \leq_{\mathrm{fin}} c \rightarrow \exists d \sqsubseteq c [a \leq_{\mathrm{fin}} d]]$.

Write down these two axioms, but don't write down **A3** and **A4**.

Axiom (**A3**, Amalgamation)

The depth function defined by, for $B \in \mathcal{R}$ and $a \in \mathcal{AR}$:

$$\mathrm{depth}_B(a) := \begin{cases} \min\{n < \omega : a \leq_{\mathrm{fin}} r_n(B)\}, & \text{if such } n \text{ exists} \\ \infty, & \text{otherwise} \end{cases}$$

satisfies the following:

(1) If $\mathrm{depth}_B(a) < \infty$, then for all $A \in [\mathrm{depth}_B(a), B]$, $[a, A] \neq \emptyset$.

(2) $A \leq B$ and $[a, A] \neq \emptyset$ imply that there exists $A' \in [\mathrm{depth}_B(a), B]$ such that $\emptyset \neq [a, A'] \subseteq [a, A]$.

Axiom (A3, Amalgamation)

The depth function defined by, for $B \in \mathcal{R}$ and $a \in \mathcal{AR}$:

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This is hard to explain, so don't spend too much time on this slide. Just briefly tell them that it allows us to do some "stacking" of arguments. We won't be using it in the talk anyway.

We let $\mathcal{AR}_n$ be the image of the map $r_n(-)$, i.e. the set of all finite approximations of length n.

Axiom (A4, Pigeonhole)

If $\text{depth}_B(a) < \infty$ and if $\mathcal{O} \subseteq \mathcal{AR}_{\text{lh}(a)+1}$, then there exists $A \in [\text{depth}_B(a), B]$ such that $r_{\text{lh}(a)+1}[a, A] \subseteq \mathcal{O}$ or $r_{\text{lh}(a)+1}[a, A] \subseteq \mathcal{O}^c$.

It's easy to check that $([\mathbb{N}]^\infty, \subseteq, r)$ satisfies the axioms **A1**-**A4**.

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It's easy to check that $([\mathbb{N}]^\infty, \subseteq, r)$ satisfies the axioms **A1**-**A4**.

Explain that for $([\mathbb{N}]^\infty, \subseteq, r)$, pigeonhole is satisfied because if $A \in [\mathbb{N}]^\infty$ and $B \subseteq A$, then either B is infinite or $A \setminus B$ is infinite. Another way to see it is this: There isn't an infinite set $A \in [\mathbb{N}]^\infty$ that intersects every infinite set B and its complement B^c.

Observe that every $A \in \mathcal{R}$ can be uniquely identified as the sequence $(r_n(A))_{n<\omega}$, an element of $\mathcal{AR}^{\mathbb{N}}$. Therefore, we may identify $\mathcal{R}$ as a subset of $\mathcal{AR}^{\mathbb{N}}$.

Definition

$(\mathcal{R}, \leq, r)$ is a *closed triple* if for all $\sqsubseteq$-increasing sequence $(a_n)_{n<\omega}$ of elements in $\mathcal{AR}$ such that $\mathrm{lh}(a_n) = n$, there exists some $A \in \mathcal{R}$ such that $r_n(A) = a_n$ for all n.

In other words, $\mathcal{R}$ is a metrically closed subset of $\mathcal{AR}^{\mathbb{N}}$.

It turns out that there is a rich Ramsey theory around closed triples $(\mathcal{R}, \leq, r)$ satisfying **A1**-**A4**. Todorčević showed that all such triples are in fact *topological Ramsey spaces*.

Definition

Let $(\mathcal{R} \leq, r)$ be a closed triple satisfying **A1**-**A4**. A set $\mathcal{X} \subseteq \mathcal{R}$ is *Ramsey* if for all $A \in \mathcal{R}$ and $a \in \mathcal{AR}$ such that $[a, A] \neq \emptyset$, there exists some $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \subseteq \mathcal{X}^c$.

Example (Countable Vector Spaces)

Let $\mathbb{F}$ be a countable field. We fix a vector space E over $\mathbb{F}$ with a countable Hamel basis $(e_n)_{n<\omega}$. For each $v \in E$, if $v = \sum_{n<\omega} a_n e_n$, then define $\operatorname{supp}(v) := \{n < \omega : a_n \neq 0\}$. We then define a partial order $<$ by $v < w$ iff $\operatorname{supp}(v) < \operatorname{supp}(w)$.

1. Let $\mathcal{R} = E^{[\infty]}$ be the set of $<$-increasing vectors.
2. Write $A \leq B$ iff $\operatorname{span}(A) \subseteq \operatorname{span}(B)$.
3. Let $r_n((x_0, x_1, \ldots)) := (x_0, \ldots, x_{n-1})$.

Then $(E^{[\infty]}, \leq, r)$ satisfies **A1**, **A3**, and **A2** with "finite" replaced with "countable". Furthermore:

- $(E^{[\infty]}, \leq, r)$ satisfies **A4** iff $|F| = 2$.
- $(E^{[\infty]}, \leq, r)$ satisfies **A2** iff $|F| < \aleph_0$.

Although countable vector spaces do not always satisfy **A2** and **A4**, Ramsey theory of vector spaces remains a well-studied topic.

Axiom (**wA2**, Weak Finitisation)

There is a quasi-ordering $\leq_{\mathrm{fin}}$ on $\mathcal{AR}$ such that:

(w1) $\{a \in \mathcal{AR} : a \leq_{\mathrm{fin}} b\}$ is countable for all $b \in \mathcal{AR}$.

(2) $A \leq B$ iff $\forall n \exists m [r_n(A) \leq_{\mathrm{fin}} r_m(B)]$.

(3) $\forall a, b \in \mathcal{AR}[a \sqsubseteq b \wedge b \leq_{\mathrm{fin}} c \rightarrow \exists d \sqsubseteq c[a \leq_{\mathrm{fin}} d]]$.

Definition

A triple $(\mathcal{R}, \leq, r)$ is a *weak* **A2** *space*, or just **wA2**-*space*, if it is a closed triple satisfying **A1**, **wA2**, **A3**.

All closed triples satisfying **A1**-**A4** are **wA2**-spaces, and countable vector spaces are **wA2**-spaces.

Definition

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All closed triples satisfying **A1**-**A4** are **wA2**-spaces, and countable vector spaces are **wA2**-spaces.

Spend a bit more time on this slide. Draw out a line of the e_n's to give a picture of what support is, and what the partial order $<$ looks like.

Abstract Kastanas Game

The abstract Kastanas game was originally considered by Cano-Di Prisco on closed triples satisfying **A1**-**A4**, but the same definition makes sense for **wA2**-spaces.

Definition (Cano, Di Prisco)

Let $(\mathcal{R}, \leq, r)$ be a **wA2**-space. Let $A \in \mathcal{R}$ and $a \in \mathcal{AR}$ be such that $[a, A] \neq \emptyset$. The *Kastanas game* played below $[a, A]$, denoted as $K[a, A]$, is:

I	$A_0 \in [a, A]$		$A_1 \in [a_1, B_0]$	$\cdots$
II		$a_1 \in r_{\mathrm{lh}(a)+1}[a, A_0]$ $B_0 \in r_{\mathrm{lh}(a_1)+1}[a_1, A_0]$		$a_2 \in [a_1, A_1]$ $B_1 \in [a_2, A_1]$

The outcome of this game is $\lim_{n \to \infty} a_n$, i.e. the unique $B \in \mathcal{R}$ such that $r_{\mathrm{lh}(a)+n}(B) = a_n$ for all n.

Definition

Let $(\mathcal{R} \leq, r)$ be a **wA2**-space. A set $\mathcal{X} \subseteq \mathcal{R}$ is *Kastanas Ramsey* if for all $A \in \mathcal{R}$ and $a \in \mathcal{AR}$ such that $[a, A] \neq \emptyset$, there exists some $B \in [a, A]$ such that one of the following holds:

1. **I** has a strategy in $K[a, B]$ to reach $\mathcal{X}^c$.
2. **II** has a strategy in $K[a, B]$ to reach $\mathcal{X}$.

Theorem (Cano, Di Prisco)

If $(\mathcal{R}, \leq, r)$ is a closed triple satisfying **A1**-**A4**, *and $\mathcal{R}$ is selective, then $\mathcal{X}$ is Ramsey iff it is Kastanas Ramsey.*

Theorem (Y.)

If $(\mathcal{R}, \leq, r)$ is a closed triple satisfying **A1**-**A4**, *then $\mathcal{X}$ is Ramsey iff it is Kastanas Ramsey.*

Set-theoretic properties of Kastanas Ramsey sets

We recap some classical results about Ramsey subsets about Ramsey subsets of $[\mathbb{N}]^\infty$.

Theorem (Galvin-Prikry)

Borel subsets of $[\mathbb{N}]^\infty$ are Ramsey.

Theorem (Silver)

Analytic subsets of $[\mathbb{N}]^\infty$ are Ramsey.

Note that if $\mathcal{AR}$ is countable, then $(\mathcal{R}, \leq, r)$ may be endowed with a Polish topology.

Theorem (Todorčević)

Let $(\mathcal{R}, \leq, r)$ be a closed triple satisfying **A1**-**A4**, *and assume that $\mathcal{AR}$ is countable. Then every analytic subsets of $\mathcal{R}$ is Ramsey.*

What about Kastanas Ramsey sets?

By Borel determinacy for $\mathbb{R}^\omega$, if $(\mathcal{R}, \leq, r)$ is a **wA2**-space and $\mathcal{AR}$ is countable, then every Borel subset of $\mathcal{R}$ is Kastanas Ramsey.

Theorem (Y.)

Let $(\mathcal{R}, \leq, r)$ be a **wA2**-space, and assume that $\mathcal{AR}$ is countable. Then every analytic subset of $\mathcal{R}$ is Kastanas Ramsey.

Combined with the earlier proposition, this generalises Todorčević's theorem above. We will spend the rest of today's talk proving this result.

Constructing **wA2**-spaces

Let $(\mathcal{R}, \leq, r)$ be a **wA2**-space. We shall construct another **wA2**-space $(\mathcal{R} \times 2^\omega, \preceq, r)$ as follows:

1. $\mathcal{A}(\mathcal{R} \times 2^\omega) := \bigcup_{n<\omega} \mathcal{AR}_n \times 2^n$.
2. Given $(A, u) \in \mathcal{R} \times 2^\omega$, let $r_n(A, x) := (r_n(A), u{\restriction}n)$.
3. We define a $\preceq_{\text{fin}}$ on $\mathcal{A}(\mathcal{R} \times 2^\omega)$ by stipulating that $(a, p) \preceq_{\text{fin}} (b, q)$ iff $a \leq_{\text{fin}} b$.
4. Given $(A, u), (B, v) \in \mathcal{R} \times 2^\omega$, we write:

$$(A, u) \preceq (B, v) \iff \forall n \exists m [r_n(A, u) \preceq_{\text{fin}} (B, v)]$$

Note that $\preceq$ is not a partial order.

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$$(A, u) \preceq (B, v) \iff \forall n \exists m [r_n(A, u) \preceq_{\mathrm{fin}} (B, v)]$$

Note that $\preceq$ is not a partial order.

Write out an example by taking $(x_0, x_1, \dots) \in \mathcal{R}$ and illustrate the various points there.

Lemma

Let $(\mathcal{R}, \leq, r)$ be a **wA2***-space. Then the closed triple $(\mathcal{R} \times 2^\omega, \preceq, r)$ defined above is a* **wA2***-space which does not satisfy* **A4***.*

This lemma is very easy to verify, but since I didn't explain the axioms, I will not give a proof.

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Let $(\mathcal{R}, \leq, r)$ be a **wA2***-space. Then the closed triple $(\mathcal{R} \times 2^\omega, \preceq, r)$ defined above is a* **wA2***-space which does not satisfy* **A4**.

This lemma is very easy to verify, but since I didn't explain the axioms, I will not give a proof.

Emphasise that this is the power of **wA2**-space. If you require all axioms **A1**-**A4** to hold, then they're restrictive and it is difficult to construct another topological Ramsey space from one. On the other hand, if you study a specific example like countable vector space, then it's too restrictive in the sense that $E^{[\infty]} \times 2^\omega$ is no longer a vector space.

Let $\pi_0 : \mathcal{R} \times 2^\omega \to \mathcal{R}$ be the projection to the first coordinate.

Proposition

Let $(\mathcal{R}, \leq, r)$ be a **wA2**-space. If $\mathcal{C} \subseteq \mathcal{R} \times 2^\omega$ is Kastanas Ramsey, then $\pi_0[\mathcal{C}] \subseteq \mathcal{R}$ is Kastanas Ramsey.

Proof of Theorem.

Let $\mathcal{X} \subseteq \mathcal{R}$ be analytic. Then there exists a Borel set (in fact, a G_δ set) $\mathcal{C} \subseteq \mathcal{R} \times 2^\omega$ such that $\mathcal{X} = \pi_0[\mathcal{R}]$. Since $\mathcal{C}$ is Borel, it is Kastanas Ramsey. By the above proposition, $\mathcal{X}$ is Kastanas Ramsey. □

We split the proof of the proposition into two lemmas.

Lemma

Let $(\mathcal{R}, \leq, r)$ be a **wA2***-space. Let $\mathcal{C} \subseteq \mathcal{R} \times 2^\omega$ be a subset. Let $A \in \mathcal{R}$ $a \in \mathcal{AR}$ such that $[a, A] \neq \emptyset$. If* **II** *has a strategy in $K[(a, p), (A, \vec{0})]$ to reach $\mathcal{C}$ for some $p \in 2^{\mathrm{lh}(a)}$, then* **II** *has a strategy in $K[a, A]$ to reach $\pi_0[\mathcal{C}]$.*

We split the proof of the proposition into two lemmas.

Lemma

Let $(\mathcal{R}, \leq, r)$ be a **wA2***-space. Let $\mathcal{C} \subseteq \mathcal{R} \times 2^\omega$ be a subset. Let $A \in \mathcal{R}$ $a \in \mathcal{AR}$ such that $[a, A] \neq \emptyset$. If* **II** *has a strategy in $K[(a, p), (A, \vec{0})]$ to reach $\mathcal{C}$ for some $p \in 2^{\mathrm{lh}(a)}$, then* **II** *has a strategy in $K[a, A]$ to reach $\pi_0[\mathcal{C}]$.*

Compare the two Kastanas game in $\mathcal{R} \times 2^\omega$ and in $\mathcal{R}$. First draw out the table for $\mathcal{R}$, follows by the table for $\mathcal{R} \times 2^\omega$. Then explain that if **II** can complete the $\mathcal{R} \times 2^\omega$ game using σ, then projecting the way **II** plays the game to the $\mathcal{R}$ game would be a good strategy. (Draw out the play one by one!)

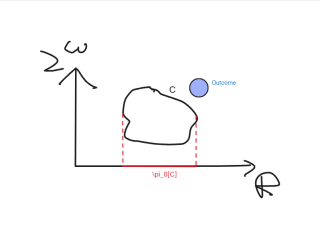
Lemma

Let $(\mathcal{R}, \leq, r)$ be a **wA2***-space. Let $\mathcal{C} \subseteq \mathcal{R} \times 2^\omega$ be a subset. Let $A \in \mathcal{R}$ and $a \in \mathcal{AR}$ such that $[a, A] \neq \emptyset$. If for all $p \in 2^{\mathrm{lh}(a)}$ and $B \in [a, A]$, there exists some $C \in [a, B]$ such that* **I** *has a strategy in $K[(a, p), (C, \vec{0})]$ to reach $\mathcal{C}^c$, then* **I** *has a strategy in $K[a, A]$ to reach $\pi_0[\mathcal{C}]^c$.*

Lemma

Let $(\mathcal{R}, \leq, r)$ be a **wA2***-space. Let $\mathcal{C} \subseteq \mathcal{R} \times 2^\omega$ be a subset. Let $A \in \mathcal{R}$ and $a \in \mathcal{AR}$ such that $[a, A] \neq \emptyset$. If for all $p \in 2^{\mathrm{lh}(a)}$ and $B \in [a, A]$, there exists some $C \in [a, B]$ such that* **I** *has a strategy in $K[(a, p), (C, \vec{0})]$ to reach $\mathcal{C}^c$, then* **I** *has a strategy in $K[a, A]$ to reach $\pi_0[\mathcal{C}]^c$.*

First explain why the same proof doesn't work. Draw this diagram:



Let's suppose that **I** has a strategy σ in $K[(a, p), (A, \vec{0})]$ to reach $\mathcal{C}^c$. With **I** following the strategy, we get the outcome to be $(B, u) \in \mathcal{C}^c$ for some $u \in 2^\omega$. However, this does not imply that $\pi_0[\mathcal{C}]^c$, as it's possible that $(B, v) \in \mathcal{C}$ for some other $v \in 2^\omega$.

To overcome this problem, we have to "stack" the strategies for all possible $u \in 2^\omega$.

Lemma

Let $(\mathcal{R}, \leq, r)$ be a **wA2***-space. Let $\mathcal{C} \subseteq \mathcal{R} \times 2^\omega$ be a subset. Let $A \in \mathcal{R}$ and $a \in \mathcal{AR}$ such that $[a, A] \neq \emptyset$. If for all $p \in 2^{\mathrm{lh}(a)}$ and $B \in [a, A]$, there exists some $C \in [a, B]$ such that* **I** *has a strategy in $K[(a, p), (C, \vec{0})]$ to reach $\mathcal{C}^c$, then* **I** *has a strategy in $K[a, A]$ to reach $\pi_0[\mathcal{C}]^c$.*

Assume for simplicity that $a = \emptyset$.
Let's say σ is a strategy for **I** in $\mathcal{R} \times 2^\omega$ game. Explain the proof in the following manner:

1. Draw out the game table for the $\mathcal{R}$ game, with just the first move by **I**. Do the same for $\mathcal{R} \times 2^\omega$.
2. Write out a response by **II** in $\mathcal{R}$ game. Use arrows to split the $\mathcal{R} \times 2^\omega$ table into two possible cases: $(a_1, (0))$ and $(a_1, (1))$. Explain how to stack the strategy from the (0) case followed by (1) case. The result is the next move.
3. Split it into four tables now, and repeat it. This should be enough.

Gowers game and asymptotic game

In 2005, Gowers resolved a long-standing Banach space theory problem with the following theorem.

Theorem (Gowers)

Every Banach space has a subspace which either has an unconditional basis or is hereditarily indecomposable.

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Start with that "the motivation for considering games on countable vector spaces started with Gowers' paper...".

Ramsey theory was used heavily in his proof. In particular, he was interested in the following games.

Definition (Gowers game)

The *Gowers game* played below $[a, A]$, denoted as $G[a, A]$, is the following game:

I	$A_0 \le A$		$A_1 \le A$		$\cdots$
II		$x_0 \in \langle A_0 \rangle$		$x_1 \in \langle A_1 \rangle$	$\cdots$

The outcome of this game is the sequence $a^\frown (x_k)_{k<\omega} \in E^{[\infty]}$.

This led to a study of similar variants of the Gowers game. The following variant was considered by Rosendal.

Definition (Asymptotic game)

The *asymptotic game* played below $[a, A]$, denoted as $F[a, A]$, is the following game:

I	A/n_0		A/n_1		$\cdots$
II		$x_0 \in \langle A/n_0 \rangle$		$x_1 \in \langle A/n_1 \rangle$	$\cdots$

The outcome of this game is the sequence $a^\frown(x_k)_{k<\omega} \in E^{[\infty]}$.

Definition

A subset $\mathcal{X} \subseteq E^{[\infty]}$ is *strategically Ramsey* if for all $A \in E^{[\infty]}$ and $a \in E^{[<\infty]}$, there exists some $B \leq A$ such that one of the following holds:

1. **I** has a strategy in $F[a, B]$ to reach $\mathcal{X}^c$,
2. **II** has a strategy in $G[a, B]$ to reach $\mathcal{X}$,

Theorem (Rosendal)

Every analytic subset of $E^{[\infty]}$ is strategically Ramsey.

Proposition

A subset $\mathcal{X} \subseteq E^{[\infty]}$ is strategically Ramsey iff it is Kastanas Ramsey.

We shall give a sketch of the proof of this theorem for the remainder of today's talk.

Two more games

The following game was also considered by Rosendal.

Definition (Adversarial Gowers game)

The *adversarial Gowers game* played below $[a, A]$, denoted as $AG[a, A]$, is the following game:

$$
\begin{array}{c|ccccc}
\textbf{I} & & & y_0 \in \langle B_0 \rangle & & \cdots \\
 & A_0 \leq A & & A_1 \leq A & & \cdots \\
\hline
\textbf{II} & & x_0 \in \langle A_0 \rangle & & x_1 \in \langle A_1 \rangle & \cdots \\
 & & B_0 \leq A & & B_1 \leq A & \cdots
\end{array}
$$

The outcome of this game is the sequence $a^\frown(x_0, y_0, x_1, \dots) \in E^{[\infty]}$.

Definition

The game $AG_{\mathbf{I}}[a, A]$, is:

I	$A/n_0 \leq A$		$y_0 \in \langle B_0 \rangle$ $A/n_1 \leq A$		$\cdots$
II		$x_0 \in \langle A/n_0 \rangle$ $B_0 \leq A$		$x_1 \in \langle A/n_1 \rangle$ $B_1 \leq A$	$\cdots$

The outcome is $a^\frown(x_0, y_0, x_1, \dots)$.

Definition

The game $AG_{\mathbf{II}}[a, A]$, is:

I	$A_0 \leq A$		$y_0 \in \langle A/n_0 \rangle$ $A_1 \leq A$		$\cdots$
II		$x_0 \in \langle A_0 \rangle$ $A/n_0 \leq A$		$x_1 \in \langle A_1 \rangle$ $A/n_1 \leq A$	$\cdots$

The outcome is $a^\frown(x_0, y_0, x_1, \dots)$.

Definition

A set $\mathcal{X} \subseteq E^{[\infty]}$ is *adversarially Ramsey* if for all $A \in E^{[\infty]}$ and $a \in E^{[<\infty]}$, there exists some $B \leq A$ such that one of the following holds:

1. **I** has a strategy in $AG_{\mathbf{I}}[a, B]$ to reach $\mathcal{X}^c$.
2. **II** has a strategy in $AG_{\mathbf{II}}[a, B]$ to reach $\mathcal{X}$.

The following game was considered by de Rancourt.

Definition (de Rancourt game)

The *adversarial Gowers game* played below $[a, A]$, denoted as $R[a, A]$, is the following game:

I	$A_0 \leq A$		$y_0 \in \langle B_0 \rangle$ $A_1 \leq B_0$		$\cdots$ $\cdots$
II		$x_0 \in \langle A_0 \rangle$ $B_0 \leq A_0$		$x_1 \in \langle A_1 \rangle$ $B_1 \leq A_1$	$\cdots$ $\cdots$

The outcome is $a^\frown(x_0, y_0, x_1, \dots) \in E^{[\infty]}$.

Definition

A set $\mathcal{X} \subseteq E^{[\infty]}$ is *de Rancourt Ramsey* if for all $A \in E^{[\infty]}$ and $a \in E^{[<\infty]}$, there exists some $B \leq A$ such that one of the following holds:

1. **I** has a strategy in $R[a, B]$ to reach $\mathcal{X}^c$.
2. **II** has a strategy in $R[a, B]$ to reach $\mathcal{X}$.

Theorem (de Rancourt)

A subset $\mathcal{X} \subseteq E^{[\infty]}$ is adversarially Ramsey iff it is de Rancourt Ramsey.

Definition

A set $\mathcal{X} \subseteq E^{[\infty]}$ is *de Rancourt Ramsey* if for all $A \in E^{[\infty]}$ and $a \in E^{[<\infty]}$, there exists some $B \leq A$ such that one of the following holds:

1. **I** has a strategy in $R[a, B]$ to reach $\mathcal{X}^c$.
2. **II** has a strategy in $R[a, B]$ to reach $\mathcal{X}$.

Theorem (de Rancourt)

A subset $\mathcal{X} \subseteq E^{[\infty]}$ is adversarially Ramsey iff it is de Rancourt Ramsey.

Write out the definitions of adversarially Ramsey and de Rancourt Ramsey for comparison.

Kastanas Ramsey $\iff$ strategically Ramsey

For simplicity, we prove for the case of $a = \emptyset$.

de Rancourt's approach was the following: For every block sequence A, he proved the following:

1. **I** has a strategy in $AG_{\mathbf{I}}[B]$ for some $B \leq A$ to ensure that the outcome is in $\mathcal{X}^c$, iff **I** has a strategy in $AG_{\mathbf{I}}[B]$ for some $B \leq A$ to ensure that the outcome is in $\mathcal{X}^c$.
2. **II** has a strategy in $AG_{\mathbf{II}}[B]$ for some $B \leq A$ to ensure that the outcome is in $\mathcal{X}$, iff **II** has a strategy in $AG_{\mathbf{II}}[B]$ for some $B \leq A$ to ensure that the outcome is in $\mathcal{X}$.

The idea is to "ignore every even coordinate".

$AG_{\mathbf{I}}[A]$:

I			$y_0 \in \langle B_0 \rangle$		$\cdots$
	$A/n_0 \leq A$		$A/n_1 \leq A$		$\cdots$
II		$x_0 \in \langle A/n_0 \rangle$		$x_1 \in \langle A/n_1 \rangle$	$\cdots$
		$B_0 \leq A$		$B_1 \leq A$	$\cdots$

$R[A]$:

I			$y_0 \in \langle B_0 \rangle$		$\cdots$
	$A_0 \leq A$		$A_1 \leq B_0$		$\cdots$
II		$x_0 \in \langle A_0 \rangle$		$x_1 \in \langle A_1 \rangle$	$\cdots$
		$B_0 \leq A_0$		$B_1 \leq A_1$	$\cdots$

The idea is to "ignore every even coordinate".

$AG_{\mathbf{I}}[A]$:

I			$y_0 \in \langle B_0 \rangle$		$\cdots$
	$A/n_0 \leq A$		$A/n_1 \leq A$		$\cdots$
II		$x_0 \in \langle A/n_0 \rangle$		$x_1 \in \langle A/n_1 \rangle$	$\cdots$
		$B_0 \leq A$		$B_1 \leq A$	$\cdots$

$R[A]$:

I			$y_0 \in \langle B_0 \rangle$		$\cdots$
	$A_0 \leq A$		$A_1 \leq B_0$		$\cdots$
II		$x_0 \in \langle A_0 \rangle$		$x_1 \in \langle A_1 \rangle$	$\cdots$
		$B_0 \leq A_0$		$B_1 \leq A_1$	$\cdots$

The idea is to "ignore every even coordinate".

$AG_{\mathbf{I}}[A]$:

I			$y_0 \in \langle B_0 \rangle$		$\cdots$
	$A/n_0 \leq A$		$A/n_1 \leq A$		$\cdots$
II		$x_0 \in \langle A/n_0 \rangle$		$x_1 \in \langle A/n_1 \rangle$	$\cdots$
		$B_0 \leq A$		$B_1 \leq A$	$\cdots$

$R[A]$:

I			$y_0 \in \langle B_0 \rangle$		$\cdots$
	$A_0 \leq A$		$A_1 \leq B_0$		$\cdots$
II		$x_0 \in \langle A_0 \rangle$		$x_1 \in \langle A_1 \rangle$	$\cdots$
		$B_0 \leq A_0$		$B_1 \leq A_1$	$\cdots$

The idea is to "ignore every even coordinate".

$AG_{\mathrm{I}}[A] \to F[A]$:

$$
\begin{array}{c|ccccc}
\mathbf{I} & & & y_0 \in \langle B_0 \rangle & & \cdots \\
 & A/n_0 \leq A & & A/n_1 \leq A & & \cdots \\
\hline
\mathbf{II} & & x_0 \in \langle A/n_0 \rangle & & x_1 \in \langle A/n_1 \rangle & \cdots \\
 & & B_0 \leq A & & B_1 \leq A & \cdots
\end{array}
$$

$R[A] \to K[A]$:

$$
\begin{array}{c|ccccc}
\mathbf{I} & & & y_0 \in \langle B_0 \rangle & & \cdots \\
 & A_0 \leq A & & A_1 \leq B_0 & & \cdots \\
\hline
\mathbf{II} & & x_0 \in \langle A_0 \rangle & & x_1 \in \langle A_1 \rangle & \cdots \\
 & & B_0 \leq A_0 & & B_1 \leq A_1 & \cdots
\end{array}
$$

The idea is to "ignore every even coordinate".

$AG_{\mathbf{II}}[A]$:

I		$y_0 \in \langle A/n_0 \rangle$		$\cdots$
	$A_0 \le A$	$A_1 \le A$		$\cdots$
II	$x_0 \in \langle A_0 \rangle$		$x_1 \in \langle A_1 \rangle$	$\cdots$
	$A/n_0 \le A$		$A/n_1 \le A$	$\cdots$

$R[A]$:

I		$y_0 \in \langle B_0 \rangle$		$\cdots$
	$A_0 \le A$	$A_1 \le B_0$		$\cdots$
II	$x_0 \in \langle A_0 \rangle$		$x_1 \in \langle A_1 \rangle$	$\cdots$
	$B_0 \le A_0$		$B_1 \le A_1$	$\cdots$

The idea is to "ignore every even coordinate".

$AG_{\mathbf{II}}[A]$:

I	$A_0 \leq A$		$y_0 \in \langle A/n_0 \rangle$ $A_1 \leq A$		$\cdots$ $\cdots$
II		$x_0 \in \langle A_0 \rangle$ $A/n_0 \leq A$		$x_1 \in \langle A_1 \rangle$ $A/n_1 \leq A$	$\cdots$ $\cdots$

$R[A]$:

I	$A_0 \leq A$		$y_0 \in \langle B_0 \rangle$ $A_1 \leq B_0$		$\cdots$ $\cdots$
II		$x_0 \in \langle A_0 \rangle$ $B_0 \leq A_0$		$x_1 \in \langle A_1 \rangle$ $B_1 \leq A_1$	$\cdots$ $\cdots$

The idea is to "ignore every even coordinate".

$AG_{\mathbf{II}}[A]$:

I			$y_0 \in \langle A/n_0 \rangle$		$\cdots$
	$A_0 \leq A$		$A_1 \leq A$		$\cdots$
II		$x_0 \in \langle A_0 \rangle$		$x_1 \in \langle A_1 \rangle$	$\cdots$
		$A/n_0 \leq A$		$A/n_1 \leq A$	$\cdots$

$R[A]$:

I			$y_0 \in \langle B_0 \rangle$		$\cdots$
	$A_0 \leq A$		$A_1 \leq B_0$		$\cdots$
II		$x_0 \in \langle A_0 \rangle$		$x_1 \in \langle A_1 \rangle$	$\cdots$
		$B_0 \leq A_0$		$B_1 \leq A_1$	$\cdots$

The idea is to "ignore every even coordinate".

$AG_{\mathbf{II}}[A] \to G[A]$:

I	$A_0 \leq A$		$y_0 \in \langle A/n_0 \rangle$ $A_1 \leq A$		$\cdots$ $\cdots$
II		$x_0 \in \langle A_0 \rangle$ $A/n_0 \leq A$		$x_1 \in \langle A_1 \rangle$ $A/n_1 \leq A$	$\cdots$ $\cdots$

$R[A] \to K[A]$:

I	$A_0 \leq A$		$y_0 \in \langle B_0 \rangle$ $A_1 \leq B_0$		$\cdots$ $\cdots$
II		$x_0 \in \langle A_0 \rangle$ $B_0 \leq A_0$		$x_1 \in \langle A_1 \rangle$ $B_1 \leq A_1$	$\cdots$ $\cdots$