There is a completely separable mad family of vector spaces

Clement Yung

13 Feb 2026

Almost disjoint subspaces

Let $\mathbb{F}$ be a countable field (possibly finite). Let E be a $\mathbb{F}$-vector space with a Hamel basis $(e_n)_{n<\omega}$.

Definition

Let $V, W \subseteq E$ be two infinite-dimensional subspaces. We say that V, W are *almost disjoint* if $V \cap W$ is a finite-dimensional subspace of E.

For the rest of this talk, the zero vector will be ignored.

Definition

Let $\mathcal{A}$ be a family of infinite-dimensional subspaces of E. We say that $\mathcal{A}$ is *almost disjoint* if all subspaces in $\mathcal{A}$ are pairwise almost disjoint. We say that $\mathcal{A}$ is *maximal almost disjoint* (or just *mad*) if $\mathcal{A}$ is not strictly contained in another almost disjoint family of infinite-dimensional subspaces.

Definition

We define the cardinal invariant:

$\mathfrak{a}_{\mathrm{vec},\mathbb{F}} := \min\{|\mathcal{A}| : \mathcal{A}$ is an infinite mad family of block subspaces$\}$.

Definition

We define the cardinal invariant:

$$\mathfrak{a}_{\mathrm{vec},\mathbb{F}} := \min\{|\mathcal{A}| : \mathcal{A} \text{ is an infinite mad family of block subspaces}\}.$$

Lemma (Smythe, 2019)

$\aleph_1 \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$.

Block subspaces

Recall that E has a fixed Hamel basis $(e_n)_{n<\omega}$. Given a vector $x \in E$, we may write

$$x = \sum_{n<\omega} \lambda_n(x) e_n,$$

where only finitely many λ_n's are non-zero.

Block subspaces

Recall that E has a fixed Hamel basis $(e_n)_{n<\omega}$. Given a vector $x \in E$, we may write

$$x = \sum_{n<\omega} \lambda_n(x) e_n,$$

where only finitely many λ_n's are non-zero. We may then write:

$$\mathrm{supp}(x) := \{n < \omega : \lambda_n(x) \neq 0\}.$$

Block subspaces

Recall that E has a fixed Hamel basis $(e_n)_{n<\omega}$. Given a vector $x \in E$, we may write

$$x = \sum_{n<\omega} \lambda_n(x) e_n,$$

where only finitely many λ_n's are non-zero. We may then write:

$$\mathrm{supp}(x) := \{n < \omega : \lambda_n(x) \neq 0\}.$$

Notation

Given two vectors x, y we write:

$$x < y \iff \max(\mathrm{supp}(x)) < \min(\mathrm{supp}(y)).$$

Definition

An infinite-dimensional subspace $V \subseteq W$ is a *block subspace* if it has a *block basis*. That is, V is spanned by a *block sequence* $(x_n)_{n<\omega}$, i.e.:

$$x_0 < x_1 < x_2 < \cdots .$$

Note that the block basis is unique up to scalar multiplication, so we may conflate block subspaces with block sequences.

Fact

Every infinite-dimensional subspace of E contains an infinite-dimensional block subspace.

Consequently, if $\mathcal{A}$ is an almost disjoint family such that there is no block subspace that is almost disjoint with every element of $\mathcal{A}$, then $\mathcal{A}$ is mad.

Notation

Let $E^{[\infty]}$ be the set of infinite block sequences of E.

Notation

If $A = (x_n)_{n<\omega} \in E^{[\infty]}$, we write:

$$\langle A \rangle := \langle x_n : n < \omega \rangle = \operatorname{span}\{x_n : n < \omega\}.$$

Completely separable mad families (on ω)

Let $\mathcal{A}$ be an almost disjoint family on ω. The collection:

$$\mathcal{I}(\mathcal{A}) := \left\{ X \subseteq \omega : X \subseteq^* \bigcup \mathcal{F} \text{ for some finite } \mathcal{F} \subseteq \mathcal{A} \right\}.$$

is an ideal.

Completely separable mad families (on ω)

Let $\mathcal{A}$ be an almost disjoint family on ω. The collection:

$$\mathcal{I}(\mathcal{A}) := \left\{ X \subseteq \omega : X \subseteq^* \bigcup \mathcal{F} \text{ for some finite } \mathcal{F} \subseteq \mathcal{A} \right\}.$$

is an ideal. We also let:

$$\begin{aligned} \mathcal{I}^+(\mathcal{A}) &:= [\omega]^\omega \setminus \mathcal{I}(\mathcal{A}), \\ \mathcal{I}^{++}(\mathcal{A}) &:= \{X \subseteq \omega : X \cap A \text{ is infinite for infinitely many } A \in \mathcal{A}\}. \end{aligned}$$

Note that $\mathcal{I}^{++}(\mathcal{A}) \subseteq \mathcal{I}^+(\mathcal{A})$, and $\mathcal{I}^{++}(\mathcal{A}) = \mathcal{I}^+(\mathcal{A})$ if $\mathcal{A}$ is mad.

Definition

Let $\mathcal{A}$ be an infinite almost disjoint family on ω. We say that $\mathcal{A}$ is *completely separable* if for all $X \in \mathcal{I}^+(\mathcal{A})$, there exists some $A \in \mathcal{A}$ such that $A \subseteq X$.

Note that every completely separable almost disjoint family is mad.

Completely separable mad families on ω can be constructed under the following assumptions:

Completely separable mad families on ω can be constructed under the following assumptions:

1. (Balcar-Simon, 1989) $\mathfrak{a} = \mathfrak{c}$ or $\mathfrak{b} = \mathfrak{d}$ or $\mathfrak{s} = \omega_1$.

Completely separable mad families on ω can be constructed under the following assumptions:

1. (Balcar-Simon, 1989) $\mathfrak{a} = \mathfrak{c}$ or $\mathfrak{b} = \mathfrak{d}$ or $\mathfrak{s} = \omega_1$.
2. (Simon, 1996) $\mathfrak{d} \leq \mathfrak{a}$.

Completely separable mad families on ω can be constructed under the following assumptions:

1. (Balcar-Simon, 1989) $\mathfrak{a} = \mathfrak{c}$ or $\mathfrak{b} = \mathfrak{d}$ or $\mathfrak{s} = \omega_1$.
2. (Simon, 1996) $\mathfrak{d} \leq \mathfrak{a}$.
3. (Shelah, 2011) $\mathfrak{s} < \mathfrak{a}$, or ($\mathfrak{a} \leq \mathfrak{s}$ + a certain PCF hypothesis).
    - For instance, $\mathfrak{c} < \aleph_\omega$ satisfies this PCF hypothesis.

Completely separable mad families on ω can be constructed under the following assumptions:

1. (Balcar-Simon, 1989) $\mathfrak{a} = \mathfrak{c}$ or $\mathfrak{b} = \mathfrak{d}$ or $\mathfrak{s} = \omega_1$.
2. (Simon, 1996) $\mathfrak{d} \leq \mathfrak{a}$.
3. (Shelah, 2011) $\mathfrak{s} < \mathfrak{a}$, or ($\mathfrak{a} \leq \mathfrak{s}$ + a certain PCF hypothesis).
   - For instance, $\mathfrak{c} < \aleph_\omega$ satisfies this PCF hypothesis.
4. (Mildenberger-Raghavan-Steprāns, 2014) $\mathfrak{s} \leq \mathfrak{a}$.

Completely separable mad families on ω can be constructed under the following assumptions:

1. (Balcar-Simon, 1989) $\mathfrak{a} = \mathfrak{c}$ or $\mathfrak{b} = \mathfrak{d}$ or $\mathfrak{s} = \omega_1$.
2. (Simon, 1996) $\mathfrak{d} \leq \mathfrak{a}$.
3. (Shelah, 2011) $\mathfrak{s} < \mathfrak{a}$, or ($\mathfrak{a} \leq \mathfrak{s}$ + a certain PCF hypothesis).
    - For instance, $\mathfrak{c} < \aleph_\omega$ satisfies this PCF hypothesis.
4. (Mildenberger-Raghavan-Steprāns, 2014) $\mathfrak{s} \leq \mathfrak{a}$.

The existence of a completely separable mad family on ω in just ZFC remains open to date.

Completely separable mad families on ω can be constructed under the following assumptions:

1. (Balcar-Simon, 1989) $\mathfrak{a} = \mathfrak{c}$ or $\mathfrak{b} = \mathfrak{d}$ or $\mathfrak{s} = \omega_1$.
2. (Simon, 1996) $\mathfrak{d} \leq \mathfrak{a}$.
3. (Shelah, 2011) $\mathfrak{s} < \mathfrak{a}$, or ($\mathfrak{a} \leq \mathfrak{s}$ + a certain PCF hypothesis).
    - For instance, $\mathfrak{c} < \aleph_\omega$ satisfies this PCF hypothesis.
4. (Mildenberger-Raghavan-Steprāns, 2014) $\mathfrak{s} \leq \mathfrak{a}$.

The existence of a completely separable mad family on ω in just ZFC remains open to date.

We give a sketch of the construction of a completely seperable mad family on ω.

Construction for $\mathfrak{s} \leq \mathfrak{a}$

A important component of the proof is the notion of a *block-splitting family*.

Construction for $\mathfrak{s} \leq \mathfrak{a}$

A important component of the proof is the notion of a *block-splitting family*.

Definition

1. Let $\mathcal{P} = \{P_n : n < \omega\}$ be an interval partition. We say that a set $S \in [\omega]^\omega$ *block-splits* $\mathcal{P}$ if $\{n < \omega : P_n \subseteq S\}$ and $\{n < \omega : P_n \cap S = \emptyset\}$ are both infinite.

Construction for $\mathfrak{s} \leq \mathfrak{a}$

A important component of the proof is the notion of a *block-splitting family*.

Definition

1. Let $\mathcal{P} = \{P_n : n < \omega\}$ be an interval partition. We say that a set $S \in [\omega]^\omega$ *block-splits* $\mathcal{P}$ if $\{n < \omega : P_n \subseteq S\}$ and $\{n < \omega : P_n \cap S = \emptyset\}$ are both infinite.
2. A *block-splitting family* is a family $\mathcal{S} \subseteq [\omega]^\omega$ such that every interval partition is block-split by some $S \in \mathcal{S}$.

Construction for $\mathfrak{s} \leq \mathfrak{a}$

A important component of the proof is the notion of a *block-splitting family*.

Definition

1. Let $\mathcal{P} = \{P_n : n < \omega\}$ be an interval partition. We say that a set $S \in [\omega]^\omega$ *block-splits* $\mathcal{P}$ if $\{n < \omega : P_n \subseteq S\}$ and $\{n < \omega : P_n \cap S = \emptyset\}$ are both infinite.
2. A *block-splitting family* is a family $\mathcal{S} \subseteq [\omega]^\omega$ such that every interval partition is block-split by some $S \in \mathcal{S}$.

Lemma (Kamburelis-Weglorz, 1996)

$\max\{\mathfrak{b}, \mathfrak{s}\} = \min\{|\mathcal{S}| : \mathcal{S}$ *is a block-splitting family*$\}$.

Lemma

Let $\{S_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ *be a block-splitting family. Let* $\mathcal{A}$ *be an almost disjoint family on* ω*, and let* $X \in \mathcal{I}^+(\mathcal{A})$*. There exists some* α *such that* $X \cap S_\alpha \in \mathcal{I}^+(\mathcal{A})$ *and* $X \setminus S_\alpha \in \mathcal{I}^+(\mathcal{A})$*.*

Lemma

Let $\{S_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be a block-splitting family. Let $\mathcal{A}$ be an almost disjoint family on ω, and let $X \in \mathcal{I}^+(\mathcal{A})$. There exists some α such that $X \cap S_\alpha \in \mathcal{I}^+(\mathcal{A})$ and $X \setminus S_\alpha \in \mathcal{I}^+(\mathcal{A})$.

Proof.

If there is some $Y \subseteq X$ that is almost disjoint from every element of $\mathcal{A}$, then let S_α be any set such that $Y \cap S_\alpha$ and $Y \setminus S_\alpha$ are infinite. Then $Y \cap S_\alpha, Y \setminus S_\alpha \in \mathcal{I}^+(\mathcal{A})$.

Lemma

Let $\{S_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be a block-splitting family. Let $\mathcal{A}$ be an almost disjoint family on ω, and let $X \in \mathcal{I}^+(\mathcal{A})$. There exists some α such that $X \cap S_\alpha \in \mathcal{I}^+(\mathcal{A})$ and $X \setminus S_\alpha \in \mathcal{I}^+(\mathcal{A})$.

Proof.

If there is some $Y \subseteq X$ that is almost disjoint from every element of $\mathcal{A}$, then let S_α be any set such that $Y \cap S_\alpha$ and $Y \setminus S_\alpha$ are infinite. Then $Y \cap S_\alpha, Y \setminus S_\alpha \in \mathcal{I}^+(\mathcal{A})$.

Otherwise, there is an infinite $\{A_i : n < \omega\} \subseteq \mathcal{A}$ such that $X \cap A_i$ is infinite for all i.

Lemma

Let $\{S_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be a block-splitting family. Let $\mathcal{A}$ be an almost disjoint family on ω, and let $X \in \mathcal{I}^+(\mathcal{A})$. There exists some α such that $X \cap S_\alpha \in \mathcal{I}^+(\mathcal{A})$ and $X \setminus S_\alpha \in \mathcal{I}^+(\mathcal{A})$.

Proof.

If there is some $Y \subseteq X$ that is almost disjoint from every element of $\mathcal{A}$, then let S_α be any set such that $Y \cap S_\alpha$ and $Y \setminus S_\alpha$ are infinite. Then $Y \cap S_\alpha, Y \setminus S_\alpha \in \mathcal{I}^+(\mathcal{A})$.

Otherwise, there is an infinite $\{A_i : n < \omega\} \subseteq \mathcal{A}$ such that $X \cap A_i$ is infinite for all i. Let $\mathcal{P} = \{P_n : n < \omega\}$ be an interval partition such that $P_n \cap (X \cap A_i) \neq \emptyset$ for all $i \leq n$. Let S_α block-splits $\mathcal{P}$.

Lemma

Let $\{S_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be a block-splitting family. Let $\mathcal{A}$ be an almost disjoint family on ω, and let $X \in \mathcal{I}^+(\mathcal{A})$. There exists some α such that $X \cap S_\alpha \in \mathcal{I}^+(\mathcal{A})$ and $X \setminus S_\alpha \in \mathcal{I}^+(\mathcal{A})$.

Proof.

If there is some $Y \subseteq X$ that is almost disjoint from every element of $\mathcal{A}$, then let S_α be any set such that $Y \cap S_\alpha$ and $Y \setminus S_\alpha$ are infinite. Then $Y \cap S_\alpha, Y \setminus S_\alpha \in \mathcal{I}^+(\mathcal{A})$.

Otherwise, there is an infinite $\{A_i : n < \omega\} \subseteq \mathcal{A}$ such that $X \cap A_i$ is infinite for all i. Let $\mathcal{P} = \{P_n : n < \omega\}$ be an interval partition such that $P_n \cap (X \cap A_i) \neq \emptyset$ for all $i \leq n$. Let S_α block-splits $\mathcal{P}$.

1. $(X \cap S_\alpha) \cap A_i$ is infinite for all i.
2. $(X \setminus S_\alpha) \cap A_i$ is infinite for all i.

Thus, they're both in $\mathcal{I}^{++}(\mathcal{A}) \subseteq \mathcal{I}^+(\mathcal{A})$. ∎

Assume that $\mathfrak{s} \leq \mathfrak{a}$. Note that $\mathfrak{b} \leq \mathfrak{a}$ holds in ZFC, so we have that $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}$.

Assume that $\mathfrak{s} \leq \mathfrak{a}$. Note that $\mathfrak{b} \leq \mathfrak{a}$ holds in ZFC, so we have that $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}$. Given $X \in [\omega]^\omega$, let $X^0 := X$ and $X^1 := \omega \setminus X$. Note that:

$$X \cap Y^0 = X \cap Y,\ X \cap Y^1 = X \setminus Y.$$

Assume that $\mathfrak{s} \leq \mathfrak{a}$. Note that $\mathfrak{b} \leq \mathfrak{a}$ holds in ZFC, so we have that $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}$. Given $X \in [\omega]^\omega$, let $X^0 := X$ and $X^1 := \omega \setminus X$. Note that:

$$X \cap Y^0 = X \cap Y,\ X \cap Y^1 = X \setminus Y.$$

Let $\mathcal{S} = \{S_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be a block-splitting family. For a fixed almost disjoint family $\mathcal{A}$ and $X \in \mathcal{I}^+(\mathcal{A})$, let $\tau_X^{\mathcal{A}} \in 2^\alpha$ be the unique function where:

Assume that $\mathfrak{s} \leq \mathfrak{a}$. Note that $\mathfrak{b} \leq \mathfrak{a}$ holds in ZFC, so we have that $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}$. Given $X \in [\omega]^\omega$, let $X^0 := X$ and $X^1 := \omega \setminus X$. Note that:

$$X \cap Y^0 = X \cap Y, \; X \cap Y^1 = X \setminus Y.$$

Let $\mathcal{S} = \{S_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be a block-splitting family. For a fixed almost disjoint family $\mathcal{A}$ and $X \in \mathcal{I}^+(\mathcal{A})$, let $\tau_X^{\mathcal{A}} \in 2^\alpha$ be the unique function where:

1. α is the least ordinal such that $X \cap S_\alpha \in \mathcal{I}^+(\mathcal{A})$ and $X \setminus S_\alpha \in \mathcal{I}^+(\mathcal{A})$.

Assume that $\mathfrak{s} \leq \mathfrak{a}$. Note that $\mathfrak{b} \leq \mathfrak{a}$ holds in ZFC, so we have that $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}$. Given $X \in [\omega]^\omega$, let $X^0 := X$ and $X^1 := \omega \setminus X$. Note that:

$$X \cap Y^0 = X \cap Y,\ X \cap Y^1 = X \setminus Y.$$

Let $\mathcal{S} = \{S_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be a block-splitting family. For a fixed almost disjoint family $\mathcal{A}$ and $X \in \mathcal{I}^+(\mathcal{A})$, let $\tau_X^{\mathcal{A}} \in 2^\alpha$ be the unique function where:

1. α is the least ordinal such that $X \cap S_\alpha \in \mathcal{I}^+(\mathcal{A})$ and $X \setminus S_\alpha \in \mathcal{I}^+(\mathcal{A})$.
2. If $\beta < \alpha$, then $X \cap S_\beta^{\tau_X^{\mathcal{A}}(\beta)} \in \mathcal{I}^+(\mathcal{A})$ (so $X \cap S_\beta^{1-\tau_X^{\mathcal{A}}(\beta)} \in \mathcal{I}(\mathcal{A})$).

Observe that if $Y \subseteq X$ are in $\mathcal{I}^+(\mathcal{A})$, then $\tau_X^{\mathcal{A}} \sqsubseteq \tau_Y^{\mathcal{A}}$.

Let $\{X_\alpha : \alpha < \mathfrak{c}\}$ cofinally enumerate $[\omega]^\omega$. We recursively construct our mad family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and $\{\sigma_\alpha : \alpha < \mathfrak{c}\} \subseteq 2^{<\max\{\mathfrak{b},\mathfrak{s}\}}$ such that (where $\mathcal{A}_\alpha := \{A_\beta : \beta < \alpha\}$):

Let $\{X_\alpha : \alpha < \mathfrak{c}\}$ cofinally enumerate $[\omega]^\omega$. We recursively construct our mad family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and $\{\sigma_\alpha : \alpha < \mathfrak{c}\} \subseteq 2^{<\max\{\mathfrak{b},\mathfrak{s}\}}$ such that (where $\mathcal{A}_\alpha := \{A_\beta : \beta < \alpha\}$):

1. If $\beta < \alpha$, then $\sigma_\alpha \not\subseteq \sigma_\beta$.

Let $\{X_\alpha : \alpha < \mathfrak{c}\}$ cofinally enumerate $[\omega]^\omega$. We recursively construct our mad family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and $\{\sigma_\alpha : \alpha < \mathfrak{c}\} \subseteq 2^{<\max\{\mathfrak{b},\mathfrak{s}\}}$ such that (where $\mathcal{A}_\alpha := \{A_\beta : \beta < \alpha\}$):

1. If $\beta < \alpha$, then $\sigma_\alpha \not\sqsubseteq \sigma_\beta$.
2. If $X_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$, then $A_\alpha \subseteq X_\alpha$.

Let $\{X_\alpha : \alpha < \mathfrak{c}\}$ cofinally enumerate $[\omega]^\omega$. We recursively construct our mad family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and $\{\sigma_\alpha : \alpha < \mathfrak{c}\} \subseteq 2^{<\max\{\mathfrak{b},\mathfrak{s}\}}$ such that (where $\mathcal{A}_\alpha := \{A_\beta : \beta < \alpha\}$):

1. If $\beta < \alpha$, then $\sigma_\alpha \not\subseteq \sigma_\beta$.
2. If $X_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$, then $A_\alpha \subseteq X_\alpha$.
3. If $\xi \in \mathrm{dom}(\sigma_\alpha)$, then $A_\alpha \subseteq^* S_\xi^{\sigma_\alpha(\xi)}$.

Let $\{X_\alpha : \alpha < \mathfrak{c}\}$ cofinally enumerate $[\omega]^\omega$. We recursively construct our mad family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and $\{\sigma_\alpha : \alpha < \mathfrak{c}\} \subseteq 2^{<\max\{\mathfrak{b},\mathfrak{s}\}}$ such that (where $\mathcal{A}_\alpha := \{A_\beta : \beta < \alpha\}$):

1. If $\beta < \alpha$, then $\sigma_\alpha \not\subseteq \sigma_\beta$.
2. If $X_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$, then $A_\alpha \subseteq X_\alpha$.
3. If $\xi \in \mathrm{dom}(\sigma_\alpha)$, then $A_\alpha \subseteq^* S_\xi^{\sigma_\alpha(\xi)}$.

It's clear that if we are able to complete this construction, then $\mathcal{A}$ is completely separable.

Assume that $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ has been defined. We define $\{X_s : s \in 2^{<\omega}\} \subseteq \mathcal{I}^+(\mathcal{A}_\alpha)$ as follows:

Assume that $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ has been defined. We define $\{X_s : s \in 2^{<\omega}\} \subseteq \mathcal{I}^+(\mathcal{A}_\alpha)$ as follows:

1. $X_\emptyset := X_\alpha$ if $X_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$. Otherwise, $X_\emptyset := \omega$.

Assume that $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ has been defined. We define $\{X_s : s \in 2^{<\omega}\} \subseteq \mathcal{I}^+(\mathcal{A}_\alpha)$ as follows:

1. $X_\emptyset := X_\alpha$ if $X_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$. Otherwise, $X_\emptyset := \omega$.
2. Let $\eta_s := \tau_{X_s}^{\mathcal{A}_\alpha}$ and $\alpha_s := \mathrm{dom}(\eta_s)$.

Assume that $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ has been defined. We define $\{X_s : s \in 2^{<\omega}\} \subseteq \mathcal{I}^+(\mathcal{A}_\alpha)$ as follows:

1. $X_\emptyset := X_\alpha$ if $X_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$. Otherwise, $X_\emptyset := \omega$.
2. Let $\eta_s := \tau_{X_s}^{\mathcal{A}_\alpha}$ and $\alpha_s := \mathrm{dom}(\eta_s)$.
3. $X_{s^\frown 0} := X_s \cap S_{\alpha_s}$, $X_{s^\frown 1} := X_s \setminus S_{\alpha_s}$.

Assume that $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ has been defined. We define $\{X_s : s \in 2^{<\omega}\} \subseteq \mathcal{I}^+(\mathcal{A}_\alpha)$ as follows:

1. $X_\emptyset := X_\alpha$ if $X_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$. Otherwise, $X_\emptyset := \omega$.
2. Let $\eta_s := \tau_{X_s}^{\mathcal{A}_\alpha}$ and $\alpha_s := \mathrm{dom}(\eta_s)$.
3. $X_{s^\frown 0} := X_s \cap S_{\alpha_s}$, $X_{s^\frown 1} := X_s \setminus S_{\alpha_s}$.

Note that if $t \sqsubseteq s$, then $X_s \subseteq X_t$, so $\eta_t \sqsubseteq \eta_s$.

Assume that $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ has been defined. We define $\{X_s : s \in 2^{<\omega}\} \subseteq \mathcal{I}^+(\mathcal{A}_\alpha)$ as follows:

1. $X_\emptyset := X_\alpha$ if $X_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$. Otherwise, $X_\emptyset := \omega$.
2. Let $\eta_s := \tau_{X_s}^{\mathcal{A}_\alpha}$ and $\alpha_s := \mathrm{dom}(\eta_s)$.
3. $X_{s^\frown 0} := X_s \cap S_{\alpha_s}$, $X_{s^\frown 1} := X_s \setminus S_{\alpha_s}$.

Note that if $t \sqsubseteq s$, then $X_s \subseteq X_t$, so $\eta_t \sqsubseteq \eta_s$.

For each $f \in 2^\omega$, let $\eta_f := \bigcup_{n<\omega} \eta_{f\restriction n}$. Note that $\eta_f \in 2^{<\max\{\mathfrak{b},\mathfrak{s}\}}$, as $\mathrm{cf}(\max\{\mathfrak{b},\mathfrak{s}\}) > \omega$.

Assume that $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ has been defined. We define $\{X_s : s \in 2^{<\omega}\} \subseteq \mathcal{I}^+(\mathcal{A}_\alpha)$ as follows:

1. $X_\emptyset := X_\alpha$ if $X_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$. Otherwise, $X_\emptyset := \omega$.
2. Let $\eta_s := \tau^{\mathcal{A}_\alpha}_{X_s}$ and $\alpha_s := \mathrm{dom}(\eta_s)$.
3. $X_{s^\frown 0} := X_s \cap S_{\alpha_s}$, $X_{s^\frown 1} := X_s \setminus S_{\alpha_s}$.

Note that if $t \sqsubseteq s$, then $X_s \subseteq X_t$, so $\eta_t \sqsubseteq \eta_s$.

For each $f \in 2^\omega$, let $\eta_f := \bigcup_{n<\omega} \eta_{f\restriction n}$. Note that $\eta_f \in 2^{<\max\{\mathfrak{b},\mathfrak{s}\}}$, as $\mathrm{cf}(\max\{\mathfrak{b},\mathfrak{s}\}) > \omega$. Since $\alpha < \mathfrak{c}$, there exists some $f \in 2^\omega$ such that $\eta_f \not\sqsubseteq \sigma_\beta$ for all β. We let $\sigma_\alpha := \eta_f$.

Observe that $(X_{f \restriction n})_{n<\omega}$ forms a $\subseteq$-decreasing sequence in $\mathcal{I}^+(\mathcal{A}_\alpha)$. Let $Y \in \mathcal{I}^+(\mathcal{A}_\alpha)$ be such that $Y \subseteq^* X_{f \restriction n}$ for all n.

Observe that $(X_{f\restriction n})_{n<\omega}$ forms a $\subseteq$-decreasing sequence in $\mathcal{I}^+(\mathcal{A}_\alpha)$. Let $Y \in \mathcal{I}^+(\mathcal{A}_\alpha)$ be such that $Y \subseteq^* X_{f\restriction n}$ for all n.

Claim

There exists an almost disjoint family $\mathcal{C} \subseteq \mathcal{A}_\alpha$ such that:

1. $|\mathcal{C}| < \max\{\mathfrak{b}, \mathfrak{s}\}$.
2. *If $B \subseteq Y$ is almost disjoint from every element of $\mathcal{C}$, then B is almost disjoint from A_β for all $\beta < \alpha$.*

Now let:

$$\mathcal{C}' := \{A \cap Y : A \in \mathcal{C}\}.$$

Then $\mathcal{C}'$ is an almost disjoint family of subsets of Y.

Now let:

$$\mathcal{C}' := \{A \cap Y : A \in \mathcal{C}\}.$$

Then $\mathcal{C}'$ is an almost disjoint family of subsets of Y.

Since $|\mathcal{C}'| < \max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}$, there exists some $A_\alpha \subseteq Y$ that is almost disjoint from every element of $\mathcal{C}$. Then this A_α works.

Mad families of subspaces

We first extend our previously introduced definitions to almost disjoint families of subspaces.

Mad families of subspaces

We first extend our previously introduced definitions to almost disjoint families of subspaces.

Definition

Let $Y \subseteq E$.

1. Y is *big* if there exists some (infinite-dimensional) subspace $V \subseteq Y$.
2. Y is *small* if it is not big.
3. Y is *very small* if for all small $Z \subseteq E$, $Y \cup Z$ is small.

Note that if $\mathbb{F} \neq \mathbb{F}_2$, then finite union of small sets need not be small. In other words, small sets need not be small.

Note that if $\mathbb{F} \neq \mathbb{F}_2$, then finite union of small sets need not be small. In other words, small sets need not be small.

Example

Fix some $\mu \in \mathbb{F} \setminus \{0,1\}$. Let:

$$Y := \{x \in E : x = e_n + y \text{ for some } e_n < y\}.$$

For any subspace U and $x \in U$, $x = \lambda e_n + y$ for some $e_n < y$ and $0 \neq \lambda$. Then $\frac{1}{\lambda}x \in U \cap Y$, and $\frac{\mu}{\lambda}x \in U \cap Y^c$. Thus, Y and Y^c are small, but $E = Y \cup Y^c$ isn't.

Let $\mathcal{A}$ be an almost disjoint family of subspaces (of E). We let:

$$\mathcal{I}(\mathcal{A}) := \left\{ Y \subseteq E : Y \setminus \bigcup \mathcal{F} \text{ is small for some finite } \mathcal{F} \subseteq \mathcal{A} \right\}.$$

Let $\mathcal{A}$ be an almost disjoint family of subspaces (of E). We let:

$$\mathcal{I}(\mathcal{A}) := \left\{ Y \subseteq E : Y \setminus \bigcup \mathcal{F} \text{ is small for some finite } \mathcal{F} \subseteq \mathcal{A} \right\}.$$

We also let:

$$\mathcal{I}^{+}(\mathcal{A}) := \mathcal{P}(E) \setminus \mathcal{I}(\mathcal{A}),$$
$$\mathcal{I}^{++}(\mathcal{A}) := \{ Y \subseteq E : Y \cap V \text{ is big for infinitely many } V \in \mathcal{A} \}.$$

Let $\mathcal{A}$ be an almost disjoint family of subspaces (of E). We let:

$$\mathcal{I}(\mathcal{A}) := \left\{ Y \subseteq E : Y \setminus \bigcup \mathcal{F} \text{ is small for some finite } \mathcal{F} \subseteq \mathcal{A} \right\}.$$

We also let:

$$\begin{aligned} \mathcal{I}^{+}(\mathcal{A}) &:= \mathcal{P}(E) \setminus \mathcal{I}(\mathcal{A}), \\ \mathcal{I}^{++}(\mathcal{A}) &:= \{ Y \subseteq E : Y \cap V \text{ is big for infinitely many } V \in \mathcal{A} \}. \end{aligned}$$

Fact (Issue 1)

$\mathcal{I}(\mathcal{A})$ is "almost" an ideal.

Definition

Let $\mathcal{A}$ be an almost disjoint family of subspaces. We say that $\mathcal{A}$ is *completely separable* if for all subspaces $U \in \mathcal{I}^+(\mathcal{A})$, there exists some $V \in \mathcal{A}$ such that $V \subseteq U$.

Once again, every completely separable almost disjoint family of subspaces is maximal.

Definition

Let $\mathcal{A}$ be an almost disjoint family of subspaces. We say that $\mathcal{A}$ is *completely separable* if for all subspaces $U \in \mathcal{I}^+(\mathcal{A})$, there exists some $V \in \mathcal{A}$ such that $V \subseteq U$.

Once again, every completely separable almost disjoint family of subspaces is maximal.

Remark. The requirement for U to be a subspace cannot be removed.

Completely separable mad families of subspaces

Theorem (Y., 2025)

There is a completely separable mad family of block subspaces.

Completely separable mad families of subspaces

Theorem (Y., 2025)

There is a completely separable mad family of block subspaces.

We shall attempt to repeat the construction assuming $\mathfrak{s} \leq \mathfrak{a}$, then make the necessary adjustments along the way.

Let $\{S_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be a block-splitting family. We let:

$$Z_\alpha := \langle e_n : n \in S_\alpha \rangle .$$

Let $\{S_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be a block-splitting family. We let:

$$Z_\alpha := \langle e_n : n \in S_\alpha \rangle .$$

Lemma (Issue 2)

Let $\mathcal{A}$ be an almost disjoint family of subspace, and let $\langle B \rangle \in \mathcal{I}^+(\mathcal{A})$ be a block subspace for some $B \in E^{[\infty]}$.

1. *For all α, $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$ or $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.*
2. *If $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$, then there exists some block subspace $\langle C \rangle \in \mathcal{I}^+(\mathcal{A})$ such that $\langle C \rangle \subseteq \langle B \rangle \cap Z_\alpha$.*
3. *If $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$, then there exists some block subspace $\langle C \rangle \in \mathcal{I}^+(\mathcal{A})$ such that $\langle C \rangle \subseteq \langle B \rangle \setminus Z_\alpha$.*
4. *There exists some α such that $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$ and $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.*

One would begin by assuming that $\max\{\mathfrak{b},\mathfrak{s}\} \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$ holds, but it turns out that this is unnecessary.

Lemma (Brendle-García Ávila, 2017 + Smythe, 2019)

$\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$.

Since $\max\{\mathfrak{b},\mathfrak{s}\} \leq \mathrm{non}(\mathcal{M})$ holds in ZFC, we have that $\max\{\mathfrak{b},\mathfrak{s}\} \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$.

Let $\{Z_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be the family of block subspaces previously defined.

Let $\{Z_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be the family of block subspaces previously defined. For a fixed almost disjoint family of subspaces $\mathcal{A}$ and block subspace $\langle B \rangle \in \mathcal{I}^+(\mathcal{A})$, let $\tau_B^{\mathcal{A}} \in 2^\alpha$ be the unique function where:

Let $\{Z_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be the family of block subspaces previously defined. For a fixed almost disjoint family of subspaces $\mathcal{A}$ and block subspace $\langle B \rangle \in \mathcal{I}^+(\mathcal{A})$, let $\tau_B^{\mathcal{A}} \in 2^\alpha$ be the unique function where:

1. α is the least ordinal such that $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$ and $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.

Let $\{Z_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be the family of block subspaces previously defined. For a fixed almost disjoint family of subspaces $\mathcal{A}$ and block subspace $\langle B \rangle \in \mathcal{I}^+(\mathcal{A})$, let $\tau_B^{\mathcal{A}} \in 2^\alpha$ be the unique function where:

1. α is the least ordinal such that $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$ and $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.
2. If $\beta < \alpha$, then $\langle B \rangle \cap Z_\beta^{\tau_B^{\mathcal{A}}(\beta)} \in \mathcal{I}^+(\mathcal{A})$ (so $\langle B \rangle \cap Z_\beta^{1-\tau_B^{\mathcal{A}}(\beta)} \in \mathcal{I}(\mathcal{A})$).

Let $\{Z_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be the family of block subspaces previously defined. For a fixed almost disjoint family of subspaces $\mathcal{A}$ and block subspace $\langle B \rangle \in \mathcal{I}^+(\mathcal{A})$, let $\tau_B^{\mathcal{A}} \in 2^\alpha$ be the unique function where:

1. α is the least ordinal such that $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$ and $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.
2. If $\beta < \alpha$, then $\langle B \rangle \cap Z_\beta^{\tau_B^{\mathcal{A}}(\beta)} \in \mathcal{I}^+(\mathcal{A})$ (so $\langle B \rangle \cap Z_\beta^{1-\tau_B^{\mathcal{A}}(\beta)} \in \mathcal{I}(\mathcal{A})$).

Observe that if $\langle C \rangle \subseteq \langle B \rangle$ are in $\mathcal{I}^+(\mathcal{A})$, then $\tau_B^{\mathcal{A}} \sqsubseteq \tau_C^{\mathcal{A}}$.

Let $\{U_\alpha : \alpha < \mathfrak{c}\}$ cofinally enumerate all subspaces of E. We recursively construct our mad family of block subspaces $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and $\{\sigma_\alpha : \alpha < \mathfrak{c}\} \subseteq 2^{<\max\{\mathfrak{b},\mathfrak{s}\}}$ such that (where $\mathcal{A}_\alpha := \{A_\beta : \beta < \alpha\}$):

Let $\{U_\alpha : \alpha < \mathfrak{c}\}$ cofinally enumerate all subspaces of E. We recursively construct our mad family of block subspaces $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and $\{\sigma_\alpha : \alpha < \mathfrak{c}\} \subseteq 2^{<\max\{\mathfrak{b},\mathfrak{s}\}}$ such that (where $\mathcal{A}_\alpha := \{A_\beta : \beta < \alpha\}$):

1. If $\beta < \alpha$, then $\sigma_\alpha \not\subseteq \sigma_\beta$.

Let $\{U_\alpha : \alpha < \mathfrak{c}\}$ cofinally enumerate all subspaces of E. We recursively construct our mad family of block subspaces $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and $\{\sigma_\alpha : \alpha < \mathfrak{c}\} \subseteq 2^{<\max\{\mathfrak{b},\mathfrak{s}\}}$ such that (where $\mathcal{A}_\alpha := \{A_\beta : \beta < \alpha\}$):

1. If $\beta < \alpha$, then $\sigma_\alpha \not\subseteq \sigma_\beta$.
2. If $U_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$, then $\langle A_\alpha \rangle \subseteq U_\alpha$.

Let $\{U_\alpha : \alpha < \mathfrak{c}\}$ cofinally enumerate all subspaces of E. We recursively construct our mad family of block subspaces $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and $\{\sigma_\alpha : \alpha < \mathfrak{c}\} \subseteq 2^{<\max\{\mathfrak{b},\mathfrak{s}\}}$ such that (where $\mathcal{A}_\alpha := \{A_\beta : \beta < \alpha\}$):

1. If $\beta < \alpha$, then $\sigma_\alpha \not\sqsubseteq \sigma_\beta$.
2. If $U_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$, then $\langle A_\alpha \rangle \subseteq U_\alpha$.
3. If $\xi \in \mathrm{dom}(\sigma_\alpha)$, then $\langle A_\alpha \rangle \setminus Z_\xi^{\sigma_\alpha(\xi)}$ is small.

Let $\{U_\alpha : \alpha < \mathfrak{c}\}$ cofinally enumerate all subspaces of E. We recursively construct our mad family of block subspaces $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and $\{\sigma_\alpha : \alpha < \mathfrak{c}\} \subseteq 2^{<\max\{\mathfrak{b},\mathfrak{s}\}}$ such that (where $\mathcal{A}_\alpha := \{A_\beta : \beta < \alpha\}$):

1. If $\beta < \alpha$, then $\sigma_\alpha \not\sqsubseteq \sigma_\beta$.
2. If $U_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$, then $\langle A_\alpha \rangle \subseteq U_\alpha$.
3. If $\xi \in \mathrm{dom}(\sigma_\alpha)$, then $\langle A_\alpha \rangle \setminus Z_\xi^{\sigma_\alpha(\xi)}$ is small.

It's clear that if we are able to complete this construction, then $\mathcal{A}$ is completely separable.

Assume that $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ has been defined. We define block sequences $\{C_s : s \in 2^{<\omega}\}$ as follows:

Assume that $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ has been defined. We define block sequences $\{C_s : s \in 2^{<\omega}\}$ as follows:

1. (**Issue 3**) If $U_\alpha \in \mathcal{I}^+(\mathcal{A})$, let $\langle C_\emptyset \rangle \in \mathcal{I}^+(\mathcal{A}_\alpha)$ be a block subspace so that $\langle C_\emptyset \rangle \subseteq U_\alpha$ (otherwise, let $\langle C_\emptyset \rangle = E$).

Assume that $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ has been defined. We define block sequences $\{C_s : s \in 2^{<\omega}\}$ as follows:

1. (**Issue 3**) If $U_\alpha \in \mathcal{I}^+(\mathcal{A})$, let $\langle C_\emptyset \rangle \in \mathcal{I}^+(\mathcal{A}_\alpha)$ be a block subspace so that $\langle C_\emptyset \rangle \subseteq U_\alpha$ (otherwise, let $\langle C_\emptyset \rangle = E$).
2. Let $\eta_s := \tau_{C_s}^{\mathcal{A}_\alpha}$ and $\alpha_s := \mathrm{dom}(\eta_s)$.

Assume that $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ has been defined. We define block sequences $\{C_s : s \in 2^{<\omega}\}$ as follows:

1. (**Issue 3**) If $U_\alpha \in \mathcal{I}^+(\mathcal{A})$, let $\langle C_\emptyset \rangle \in \mathcal{I}^+(\mathcal{A}_\alpha)$ be a block subspace so that $\langle C_\emptyset \rangle \subseteq U_\alpha$ (otherwise, let $\langle C_\emptyset \rangle = E$).
2. Let $\eta_s := \tau_{C_s}^{\mathcal{A}_\alpha}$ and $\alpha_s := \mathrm{dom}(\eta_s)$.
3. $\langle C_{s\frown 0} \rangle, \langle C_{s\frown 1} \rangle \in \mathcal{I}^+(\mathcal{A})$ such that $\langle C_{s\frown 0} \rangle \subseteq \langle C_s \rangle \cap Z_{\alpha_s}$ and $\langle C_{s\frown 1} \rangle \subseteq \langle C_s \rangle \setminus Z_{\alpha_s}$.

Assume that $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ has been defined. We define block sequences $\{C_s : s \in 2^{<\omega}\}$ as follows:

1. (**Issue 3**) If $U_\alpha \in \mathcal{I}^+(\mathcal{A})$, let $\langle C_\emptyset \rangle \in \mathcal{I}^+(\mathcal{A}_\alpha)$ be a block subspace so that $\langle C_\emptyset \rangle \subseteq U_\alpha$ (otherwise, let $\langle C_\emptyset \rangle = E$).
2. Let $\eta_s := \tau_{C_s}^{\mathcal{A}_\alpha}$ and $\alpha_s := \mathrm{dom}(\eta_s)$.
3. $\langle C_{s^\frown 0} \rangle, \langle C_{s^\frown 1} \rangle \in \mathcal{I}^+(\mathcal{A})$ such that $\langle C_{s^\frown 0} \rangle \subseteq \langle C_s \rangle \cap Z_{\alpha_s}$ and $\langle C_{s^\frown 1} \rangle \subseteq \langle C_s \rangle \setminus Z_{\alpha_s}$.

Note that if $t \sqsubseteq s$, then $C_s \leq C_t$ (i.e. $\langle C_s \rangle \subseteq \langle C_t \rangle$), so $\eta_t \sqsubseteq \eta_s$.

Assume that $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ has been defined. We define block sequences $\{C_s : s \in 2^{<\omega}\}$ as follows:

1. (**Issue 3**) If $U_\alpha \in \mathcal{I}^+(\mathcal{A})$, let $\langle C_\emptyset \rangle \in \mathcal{I}^+(\mathcal{A}_\alpha)$ be a block subspace so that $\langle C_\emptyset \rangle \subseteq U_\alpha$ (otherwise, let $\langle C_\emptyset \rangle = E$).
2. Let $\eta_s := \tau^{\mathcal{A}_\alpha}_{C_s}$ and $\alpha_s := \mathrm{dom}(\eta_s)$.
3. $\langle C_{s\frown 0} \rangle, \langle C_{s\frown 1} \rangle \in \mathcal{I}^+(\mathcal{A})$ such that $\langle C_{s\frown 0} \rangle \subseteq \langle C_s \rangle \cap Z_{\alpha_s}$ and $\langle C_{s\frown 1} \rangle \subseteq \langle C_s \rangle \setminus Z_{\alpha_s}$.

Note that if $t \sqsubseteq s$, then $C_s \leq C_t$ (i.e. $\langle C_s \rangle \subseteq \langle C_t \rangle$), so $\eta_t \sqsubseteq \eta_s$.

For each $f \in 2^\omega$, let $\eta_f := \bigcup_{n<\omega} \eta_{f\restriction n}$. Note that $\eta_f \in 2^{<\max\{\mathfrak{b},\mathfrak{s}\}}$. Let $f \in 2^\omega$ be such that $\eta_f \not\sqsubseteq \sigma_\beta$ for all β. We let $\sigma_\alpha := \eta_f$.

Claim (Issue 4)

There exists some block sequence $\langle D \rangle \in \mathcal{I}^+(\mathcal{A}_\alpha)$ such that $D \leq^* C_{f \restriction n}$ for all n (i.e. $\langle D \rangle \setminus \langle C_{f \restriction n} \rangle$ is small).

Claim (Issue 4)

There exists some block sequence $\langle D \rangle \in \mathcal{I}^+(\mathcal{A}_\alpha)$ such that $D \leq^* C_{f\restriction n}$ for all n (i.e. $\langle D \rangle \setminus \langle C_{f\restriction n} \rangle$ is small).

Claim

There exists an almost disjoint family $\mathcal{C} \subseteq \mathcal{A}_\alpha$ such that:

1. $|\mathcal{C}| < \max\{\mathfrak{b}, \mathfrak{s}\}$.
2. *If $B \leq D$ is almost disjoint from every element of $\mathcal{C}$, then B is almost disjoint from A_β for all $\beta < \alpha$.*

Now let:

$$\mathcal{C}' := \{\langle A \rangle \cap \langle D \rangle : \langle A \rangle \in \mathcal{A}\}.$$

Now let:

$$\mathcal{C}' := \{\langle A\rangle \cap \langle D\rangle : \langle A\rangle \in \mathcal{A}\}.$$

Lemma

Finite intersection of block subspaces is a block subspace. Therefore, $\mathcal{C}'$ is an almost disjoint family of block subspaces of $\langle D\rangle$.

Now let:

$$\mathcal{C}' := \{\langle A\rangle \cap \langle D\rangle : \langle A\rangle \in \mathcal{A}\}.$$

Lemma

Finite intersection of block subspaces is a block subspace. Therefore, $\mathcal{C}'$ is an almost disjoint family of block subspaces of $\langle D\rangle$.

Since $|\mathcal{C}'| < \max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$, there exists some $A_\alpha \leq D$ that is almost disjoint from every element of $\mathcal{C}$. Then this A_α works.

Resolving the issues

Summary of issues:

1. Show that $\mathcal{I}(\mathcal{A})$ is "almost" an ideal.
2. The main lemma about $\{Z_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$.
3. If $U \in \mathcal{I}^+(\mathcal{A})$, then there exists some block subspace $\langle B \rangle \in \mathcal{I}^+(\mathcal{A})$ such that $\langle B \rangle \subseteq U$.
4. If $C_0 \geq C_1 \geq \cdots$ are in $\mathcal{I}^+(\mathcal{A})$, then there exists some $\langle D \rangle \in \mathcal{I}^+(\mathcal{A})$ such that $D \leq^* C_n$ for all n.

Resolving the issues

Summary of issues:

1. Show that $\mathcal{I}(\mathcal{A})$ is “almost” an ideal.
2. The main lemma about $\{Z_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$.
3. If $U \in \mathcal{I}^+(\mathcal{A})$, then there exists some block subspace $\langle B \rangle \in \mathcal{I}^+(\mathcal{A})$ such that $\langle B \rangle \subseteq U$.
4. If $C_0 \geq C_1 \geq \cdots$ are in $\mathcal{I}^+(\mathcal{A})$, then there exists some $\langle D \rangle \in \mathcal{I}^+(\mathcal{A})$ such that $D \leq^* C_n$ for all n.

Issue 1: $\mathcal{I}(\mathcal{A})$ is "almost" an ideal.

We first prove a basic lemma. Recall that a set $Y \subseteq E$ is *very small* if for all small $Z \subseteq E$, $Y \cup Z$ is small.

Lemma

Let U, V be infinite-dimensional subspaces.

1. *If $U \cap V$ is small, then $U \cap V$ is very small.*
2. *If $U \setminus V$ is small, then $U \setminus V$ is very small.*

Consequently, we cannot have both $U \cap V$ and $U \setminus V$ small.

Proof.

1. Let $Y \subseteq E$, and suppose that $W \subseteq (U \cap V) \cup Y$ for some subspace W.

Proof.

1. Let $Y \subseteq E$, and suppose that $W \subseteq (U \cap V) \cup Y$ for some subspace W. Since $U \cap V$ is a finite-dimensional subspace, let $W' \subseteq W$ be a subspace such that $W' \cap (U \cap V) = \emptyset$. Then $W' \subseteq Y$, so Y is big.

Proof.

1. Let $Y \subseteq E$, and suppose that $W \subseteq (U \cap V) \cup Y$ for some subspace W. Since $U \cap V$ is a finite-dimensional subspace, let $W' \subseteq W$ be a subspace such that $W' \cap (U \cap V) = \emptyset$. Then $W' \subseteq Y$, so Y is big.
2. Let $Y \subseteq E$, and suppose that $W \subseteq (U \setminus V) \cup Y$ for some subspace W.

Proof.

1. Let $Y \subseteq E$, and suppose that $W \subseteq (U \cap V) \cup Y$ for some subspace W. Since $U \cap V$ is a finite-dimensional subspace, let $W' \subseteq W$ be a subspace such that $W' \cap (U \cap V) = \emptyset$. Then $W' \subseteq Y$, so Y is big.
2. Let $Y \subseteq E$, and suppose that $W \subseteq (U \setminus V) \cup Y$ for some subspace W.
   2.1 If $W \cap U$ is small, then let $W' \subseteq W$ be a subspace such that $W' \cap (W \cap U) = \emptyset$. Then $W' \subseteq Y$, so Y is big.

Proof.

1. Let $Y \subseteq E$, and suppose that $W \subseteq (U \cap V) \cup Y$ for some subspace W. Since $U \cap V$ is a finite-dimensional subspace, let $W' \subseteq W$ be a subspace such that $W' \cap (U \cap V) = \emptyset$. Then $W' \subseteq Y$, so Y is big.
2. Let $Y \subseteq E$, and suppose that $W \subseteq (U \setminus V) \cup Y$ for some subspace W.
   2.1 If $W \cap U$ is small, then let $W' \subseteq W$ be a subspace such that $W' \cap (W \cap U) = \emptyset$. Then $W' \subseteq Y$, so Y is big.
   2.2 If $W \cap V$ is big, then $W \cap V \subseteq Y$, so Y is big.

Proof.

1. Let $Y \subseteq E$, and suppose that $W \subseteq (U \cap V) \cup Y$ for some subspace W. Since $U \cap V$ is a finite-dimensional subspace, let $W' \subseteq W$ be a subspace such that $W' \cap (U \cap V) = \emptyset$. Then $W' \subseteq Y$, so Y is big.
2. Let $Y \subseteq E$, and suppose that $W \subseteq (U \setminus V) \cup Y$ for some subspace W.
   - 2.1 If $W \cap U$ is small, then let $W' \subseteq W$ be a subspace such that $W' \cap (W \cap U) = \emptyset$. Then $W' \subseteq Y$, so Y is big.
   - 2.2 If $W \cap V$ is big, then $W \cap V \subseteq Y$, so Y is big.
   - 2.3 If $W \cap U$ is big but $W \cap V$ is small, then let $W' \subseteq W \cap U$ be such that $W' \cap V = \emptyset$. Then $W \subseteq U \setminus V$, a contradiction.

$\square$

Lemma

$\mathcal{I}(\mathcal{A})$ is "almost" an ideal in the following sense:

1. *$E \notin \mathcal{I}(\mathcal{A})$.*
2. *If $U \in \mathcal{I}(\mathcal{A})$, and $W \subseteq U$, then $W \in \mathcal{I}(\mathcal{A})$.*
3. *If $U, W \in \mathcal{I}(\mathcal{A})$ are two subspaces, then $U \cup W \in \mathcal{I}(\mathcal{A})$.*

Note that (2) is obvious by definition.

Proof.

1. Suppose that $E \in \mathcal{I}(\mathcal{A})$, so $E \setminus \bigcup_{i<n} V_i$ is small for some $V_0, \ldots, V_{n-1} \in \mathcal{A}$.

Proof.

1. Suppose that $E \in \mathcal{I}(\mathcal{A})$, so $E \setminus \bigcup_{i<n} V_i$ is small for some $V_0, \dots, V_{n-1} \in \mathcal{A}$. Let $V_n \in \mathcal{A} \setminus \{V_0, \dots, V_{n-1}\}$. Then:

$$V_n = \left(V_n \setminus \bigcup_{i<n} V_i \right) \cup \bigcup_{i<n} V_n \cap V_i,$$

   which is small, a contradiction.

Proof (Cont.).

3. Let $V_0, \ldots, V_{n-1} \in \mathcal{A}$ be such that $U \setminus \bigcup_{i<n} V_i$ and $W \setminus \bigcup_{i<n} V_i$ are small.

Proof (Cont.).

3. Let $V_0, \ldots, V_{n-1} \in \mathcal{A}$ be such that $U \setminus \bigcup_{i<n} V_i$ and $W \setminus \bigcup_{i<n} V_i$ are small. We shall show that $(U \cup W) \setminus \bigcup_{i<n} V_i$. Otherwise, suppose that $W' \subseteq (U \cup W) \setminus \bigcup_{i<n} V_i$.

Proof (Cont.).

3. Let $V_0, \dots, V_{n-1} \in \mathcal{A}$ be such that $U \setminus \bigcup_{i<n} V_i$ and $W \setminus \bigcup_{i<n} V_i$ are small. We shall show that $(U \cup W) \setminus \bigcup_{i<n} V_i$. Otherwise, suppose that $W' \subseteq (U \cup W) \setminus \bigcup_{i<n} V_i$. Then:

$$W' \cap U \subseteq U \setminus \bigcup_{i<n} V_i,$$

$$W' \setminus U \subseteq W \setminus \bigcup_{i<n} V_i,$$

are both small, a contradiction.

□

Issue 2: Main lemma

We recall the lemma. Let $\{S_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be a block-splitting family. We let $Z_\alpha := \langle e_n : n \in S_\alpha \rangle$.

Issue 2: Main lemma

We recall the lemma. Let $\{S_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$ be a block-splitting family. We let $Z_\alpha := \langle e_n : n \in S_\alpha \rangle$.

Lemma

Let $\mathcal{A}$ be an almost disjoint family of subspace, and let $\langle B \rangle \in \mathcal{I}^+(\mathcal{A})$ be a block subspace for some $B \in E^{[\infty]}$.

1. *For all α, $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$ or $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.*
2. *If $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$, then there exists some block subspace $\langle C \rangle \in \mathcal{I}^+(\mathcal{A})$ such that $\langle C \rangle \subseteq \langle B \rangle \cap Z_\alpha$.*
3. *If $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$, then there exists some block subspace $\langle C \rangle \in \mathcal{I}^+(\mathcal{A})$ such that $\langle C \rangle \subseteq \langle B \rangle \cap Z_\alpha$.*
4. *There exists some α such that $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$ and $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.*

Lemma

1. *For all* α, $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$ *or* $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.

Lemma

1. *For all* α, $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$ *or* $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.

Proof of (1).

Suppose for a contradiction that there are $V_0, \ldots, V_{n-1} \in \mathcal{A}$ such that $(\langle B \rangle \cap Z_\alpha) \setminus \bigcup_{i<n} V_i$ and $(\langle B \rangle \setminus Z_\alpha) \setminus \bigcup_{i<n} V_i$ are small.

Lemma

1. *For all* α, $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$ *or* $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.

Proof of (1).

Suppose for a contradiction that there are $V_0, \ldots, V_{n-1} \in \mathcal{A}$ such that $(\langle B \rangle \cap Z_\alpha) \setminus \bigcup_{i<n} V_i$ and $(\langle B \rangle \setminus Z_\alpha) \setminus \bigcup_{i<n} V_i$ are small. Since $\langle B \rangle \in \mathcal{I}^+(\mathcal{A})$, $U \subseteq \langle B \rangle \setminus \bigcup_{i<n} V_i$ for some subspace U.

Lemma

1. *For all α, $\langle B\rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$ or $\langle B\rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.*

Proof of (1).

Suppose for a contradiction that there are $V_0, \dots, V_{n-1} \in \mathcal{A}$ such that $(\langle B\rangle \cap Z_\alpha) \setminus \bigcup_{i<n} V_i$ and $(\langle B\rangle \setminus Z_\alpha) \setminus \bigcup_{i<n} V_i$ are small. Since $\langle B\rangle \in \mathcal{I}^+(\mathcal{A})$, $U \subseteq \langle B\rangle \setminus \bigcup_{i<n} V_i$ for some subspace U. Then:

$$U \cap Z_\alpha \subseteq (\langle B\rangle \cap Z_\alpha) \setminus \bigcup_{i<n} V_i,$$

$$U \setminus Z_\alpha \subseteq (\langle B\rangle \setminus Z_\alpha) \setminus \bigcup_{i<n} V_i,$$

are both small, a contradiction. □

Lemma

4. *There exists some* α *such that* $\langle B \rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$ *and* $\langle B \rangle \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.

Proof.

Suppose that there is some $C \leq B$ that is almost disjoint from every element of $\mathcal{A}$. Write $C = (x_n)_{n<\omega}$.

Proof.

Suppose that there is some $C \leq B$ that is almost disjoint from every element of $\mathcal{A}$. Write $C = (x_n)_{n<\omega}$. Let $\{P_n : n < \omega\}$ be an interval partition such that $\mathrm{supp}(x_n) \subseteq P_n$ for all n.

Proof.

Suppose that there is some $C \leq B$ that is almost disjoint from every element of $\mathcal{A}$. Write $C = (x_n)_{n<\omega}$. Let $\{P_n : n < \omega\}$ be an interval partition such that $\mathrm{supp}(x_n) \subseteq P_n$ for all n. This means that:

1. For infinitely many n, $\mathrm{supp}(x_n) \subseteq S_\alpha$, so $x_n \in Z_\alpha = \langle (e_n)_{n \in S_\alpha} \rangle$. Therefore, $C \cap Z_\alpha$ is an infinite-dimensional subspace almost disjoint from every element of $\mathcal{A}$, so $C \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.

Proof.

Suppose that there is some $C \leq B$ that is almost disjoint from every element of $\mathcal{A}$. Write $C = (x_n)_{n<\omega}$. Let $\{P_n : n < \omega\}$ be an interval partition such that $\mathrm{supp}(x_n) \subseteq P_n$ for all n. This means that:

1. For infinitely many n, $\mathrm{supp}(x_n) \subseteq S_\alpha$, so $x_n \in Z_\alpha = \langle (e_n)_{n \in S_\alpha} \rangle$. Therefore, $C \cap Z_\alpha$ is an infinite-dimensional subspace almost disjoint from every element of $\mathcal{A}$, so $C \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.
2. For infinitely many n, $\mathrm{supp}(x_n) \subseteq S_\alpha^c$. Thus:

$$x_n \in \langle (e_n)_{n \notin S_\alpha} \rangle \subseteq Z_\alpha^c.$$

Proof.

Suppose that there is some $C \leq B$ that is almost disjoint from every element of $\mathcal{A}$. Write $C = (x_n)_{n<\omega}$. Let $\{P_n : n < \omega\}$ be an interval partition such that $\operatorname{supp}(x_n) \subseteq P_n$ for all n. This means that:

1. For infinitely many n, $\operatorname{supp}(x_n) \subseteq S_\alpha$, so $x_n \in Z_\alpha = \langle (e_n)_{n \in S_\alpha} \rangle$. Therefore, $C \cap Z_\alpha$ is an infinite-dimensional subspace almost disjoint from every element of $\mathcal{A}$, so $C \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.
2. For infinitely many n, $\operatorname{supp}(x_n) \subseteq S_\alpha^c$. Thus:

$$x_n \in \langle (e_n)_{n \notin S_\alpha} \rangle \subseteq Z_\alpha^c.$$

   Therefore, $C \setminus Z_\alpha$ contains $C \cap \langle (e_n)_{n \notin S_\alpha} \rangle$, an infinite-dimensional subspace almost disjoint from every element of $\mathcal{A}$, so $C \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.

Proof (Cont.).

Suppose that for all $C \leq B$, C is not almost disjoint from some element of $\mathcal{A}$. Let $\{V_i : i < \omega\} \subseteq \mathcal{A}$ be such that $\langle B \rangle \cap V_i$ is big for all n.

Proof (Cont.).

Suppose that for all $C \leq B$, C is not almost disjoint from some element of $\mathcal{A}$. Let $\{V_i : i < \omega\} \subseteq \mathcal{A}$ be such that $\langle B \rangle \cap V_i$ is big for all n.

Fix an infinite-to-one surjection $\Gamma : \omega \to \omega$. We define a block sequence $(x_n)_{n<\omega}$ as follows:

Proof (Cont.).

Suppose that for all $C \leq B$, C is not almost disjoint from some element of $\mathcal{A}$. Let $\{V_i : i < \omega\} \subseteq \mathcal{A}$ be such that $\langle B \rangle \cap V_i$ is big for all n.

Fix an infinite-to-one surjection $\Gamma : \omega \to \omega$. We define a block sequence $(x_n)_{n<\omega}$ as follows: Let $x_0 \in \langle B \rangle \cap V_{\Gamma(0)}$.

Proof (Cont.).

Suppose that for all $C \leq B$, C is not almost disjoint from some element of $\mathcal{A}$. Let $\{V_i : i < \omega\} \subseteq \mathcal{A}$ be such that $\langle B \rangle \cap V_i$ is big for all n.

Fix an infinite-to-one surjection $\Gamma : \omega \to \omega$. We define a block sequence $(x_n)_{n<\omega}$ as follows: Let $x_0 \in \langle B \rangle \cap V_{\Gamma(0)}$. If x_n has been defined, then let $x_{n+1} \in \langle B \rangle \cap V_{\Gamma(n+1)}$ such that $x_n < x_{n+1}$.

Proof (Cont.).

We define an interval partition $\mathcal{P} = \{P_m : m < \omega\}$ as follows:

Proof (Cont.).

We define an interval partition $\mathcal{P} = \{P_m : m < \omega\}$ as follows:

1. Let $P_0 := \{0, \dots, \max(\text{supp}(x_{n_0}))\}$, where n_0 is the least integer so that $\Gamma(n_0) = 0$.

Proof (Cont.).

We define an interval partition $\mathcal{P} = \{P_m : m < \omega\}$ as follows:

1. Let $P_0 := \{0, \ldots, \max(\mathrm{supp}(x_{n_0}))\}$, where n_0 is the least integer so that $\Gamma(n_0) = 0$.
2. If P_m has been defined, then let P_{m+1} be an interval such that for all $i \leq m + 1$, there exists some $n > \max(P_m)$ such that $\Gamma(n) = i$ and $\mathrm{supp}(x_n) \subseteq P_{m+1}$.

Proof (Cont.).

We define an interval partition $\mathcal{P} = \{P_m : m < \omega\}$ as follows:

1. Let $P_0 := \{0, \dots, \max(\mathrm{supp}(x_{n_0}))\}$, where n_0 is the least integer so that $\Gamma(n_0) = 0$.
2. If P_m has been defined, then let P_{m+1} be an interval such that for all $i \leq m+1$, there exists some $n > \max(P_m)$ such that $\Gamma(n) = i$ and $\mathrm{supp}(x_n) \subseteq P_{m+1}$.

Now suppose that S_α splits $\mathcal{P}$. For each i, let:

$$T_i^0 := \{n < \omega : \Gamma(n) = i \wedge \mathrm{supp}(x_n) \subseteq S_\alpha\},$$
$$T_i^1 := \{n < \omega : \Gamma(n) = i \wedge \mathrm{supp}(x_n) \cap S_\alpha = \emptyset\}.$$

By our choice of $\mathcal{P}$, T_i^0, T_i^1 are infinite for all i.

Proof (Cont.).

We define an interval partition $\mathcal{P} = \{P_m : m < \omega\}$ as follows:

1. Let $P_0 := \{0, \dots, \max(\mathrm{supp}(x_{n_0}))\}$, where n_0 is the least integer so that $\Gamma(n_0) = 0$.
2. If P_m has been defined, then let P_{m+1} be an interval such that for all $i \leq m+1$, there exists some $n > \max(P_m)$ such that $\Gamma(n) = i$ and $\mathrm{supp}(x_n) \subseteq P_{m+1}$.

Now suppose that S_α splits $\mathcal{P}$. For each i, let:

$$T_i^0 := \{n < \omega : \Gamma(n) = i \wedge \mathrm{supp}(x_n) \subseteq S_\alpha\},$$
$$T_i^1 := \{n < \omega : \Gamma(n) = i \wedge \mathrm{supp}(x_n) \cap S_\alpha = \emptyset\}.$$

By our choice of $\mathcal{P}$, T_i^0, T_i^1 are infinite for all i. This implies that $V_i \cap (B \cap Z_\alpha)$ and $V_i \cap (B \setminus Z_\alpha)$ are big for all i, so $B \cap Z_\alpha, B \setminus Z_\alpha \in \mathcal{I}^+(\mathcal{A})$.

Thank you for listening!

1. Two (infinite-dimensional) subspaces are almost disjoint if their intersection is finite-dimensional.
2. A mad family of subspaces $\mathcal{A}$ is *completely separable* if for all subspaces $U \in \mathcal{I}^+(\mathcal{A})$, there exists some $V \in \mathcal{A}$ such that $V \subseteq U$.
3. $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$ holds in ZFC. Consequently, following the construction of a completely separable mad family on ω assuming $\mathfrak{s} \leq \mathfrak{a}$, we can show that there is a completely separable mad family of subspaces in ZFC.
4. Several issues needed to be and were addressed when replicating the construction.
   4.1 While $\mathcal{I}(\mathcal{A})$ need not be an ideal, it is "almost" an ideal.
   4.2 If $Z_\alpha := \{S_\alpha : \alpha < \max\{\mathfrak{b}, \mathfrak{s}\}\}$, then Z_α splits $\langle B \rangle \in \mathcal{I}^+(\mathcal{A})$ into two $\mathcal{I}(\mathcal{A})$-positive sets for some α.