Mad families of vector spaces

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Almost disjoint subspaces

Let $\mathbb{F}$ be a countable field (possibly finite). Let E be a $\mathbb{F}$-vector space with a Hamel basis $(e_n)_{n<\omega}$.

Definition

Let $V, W \subseteq E$ be two infinite-dimensional subspaces. We say that V, W are *almost disjoint* if $V \cap W$ is a finite-dimensional subspace of E.

Definition

Let $\mathcal{A}$ be a family of infinite-dimensional subspaces of E. We say that $\mathcal{A}$ is *almost disjoint* if all subspaces in $\mathcal{A}$ are pairwise almost disjoint. We say that $\mathcal{A}$ is *maximal almost disjoint* (or just *mad*) if $\mathcal{A}$ is not strictly contained in another almost disjoint family of infinite-dimensional subspaces.

Definition

We define the cardinal invariant:

$$\mathfrak{a}_{\mathrm{vec},\mathbb{F}} := \min\{|\mathcal{A}| : \mathcal{A} \text{ is a mad family of block subspaces}\}.$$

It's natural to ask if some properties that hold for mad families of $[\omega]^\omega$ also hold for mad families of vector spaces.

Property	$[\omega]^\omega$	Subspaces
Every mad family is uncountable	True (Easy diagonalisation)	True (Smythe, 2019)
No analytic mad family	True (Mathias, 1977)	Mostly open, partial results
Relationship between $\mathfrak{a}$ and $\mathfrak{a}_{\mathrm{vec},\mathbb{F}}$	$\mathfrak{a} < \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$ is consistent (Smythe et al., 2019) $\mathfrak{a} > \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$ is open	

Block subspaces

Recall that E has a fixed Hamel basis $(e_n)_{n<\omega}$. Given a vector $x \in E$, we may write

$$x = \sum_{n<\omega} \lambda_n(x) e_n,$$

where only finitely many λ_n's are non-zero. We may then write:

$$\mathrm{supp}(x) := \{n < \omega : \lambda_n(x) \neq 0\}.$$

Example

If $x = 2e_3 - 6e_{17} + 5e_{58}$, then $\mathrm{supp}(x) = \{3, 17, 58\}$.

Notation

Given two vectors x, y we write:

$$x < y \iff \max(\mathrm{supp}(x)) < \min(\mathrm{supp}(y)).$$

Example

If:

1. $x = 2e_3 - 6e_{17} + 5e_{58}$,
2. $y = 5e_{67} + 990e_{133} - 155e_{236}$,
3. $z = -32e_{43} + 5e_{665}$,

then $x < y$ but $x \not< z$.

Definition

An infinite-dimensional subspace $V \subseteq W$ is a *block subspace* if it has a (unique) *block basis*. That is, V is spanned by the basis $(x_n)_{n<\omega}$, where:

$$x_0 < x_1 < x_2 < \cdots .$$

Fact

Every infinite-dimensional subspace of E contains an infinite-dimensional block subspace.

Consequently, if $\mathcal{A}$ is an almost disjoint family such that there is no block subspace that is almost disjoint with every element of $\mathcal{A}$, then $\mathcal{A}$ is mad.

Notation

Let $E^{[\infty]}$ denote the set of block sequences (i.e. block bases) of E. That is, the set of sequences $(x_n)_{n<\omega}$ such that $x_0 < x_1 < x_2 < \cdots$. If $A = (x_n)_{n<\omega} \in E^{[\infty]}$, we write:

$$\langle A \rangle = \langle x_n : n < \omega \rangle := \mathrm{span}\{x_n : n < \omega\}.$$

Uncountability of mad families

Proposition (Smythe, 2019)

Every mad family $\mathcal{A} \subseteq E^{[\infty]}$ is uncountable.

The key lemma is the following:

Lemma

Let $A \in E^{[\infty]}$, and let $x_0, \ldots, x_n$ be non-zero vectors. Then there exists some M such that for any $x \notin \langle A \rangle$ such that whenever $x > M$ (i.e. $\min(\operatorname{supp}(x)) > M$),

$$\langle x_0, \ldots, x_n, x \rangle \cap \langle A \rangle = \langle x_0, \ldots, x_n \rangle \cap \langle A \rangle .$$

Proof for the $[\omega]^\omega$ case. Suppose that $\mathcal{A} = \{A_n : n < \omega\} \subseteq [\omega]^\omega$ is almost disjoint.

Step 1. Choose any $x_0 \in A_0$, so that:

$$\{x_0\} \cap A_0 \subseteq \{x_0\}.$$

Step 2. Choose $x_1 \in A_1$ large enough, so that:

$$\{x_0, x_1\} \cap A_0 \subseteq \{x_0\},$$
$$\{x_0, x_1\} \cap A_1 \subseteq \{x_0, x_1\}.$$

Step 3. Choose $x_2 \in A_2$ large enough, so that:

$$\{x_0, x_1, x_2\} \cap A_0 \subseteq \{x_0\},$$
$$\{x_0, x_1, x_2\} \cap A_1 \subseteq \{x_0, x_1\},$$
$$\{x_0, x_1, x_2\} \cap A_2 \subseteq \{x_0, x_1, x_2\}.$$

and so on. Then $\{x_n : n < \omega\}$ is almost disjoint from $\mathcal{A}$.

Proof for the $E^{[\infty]}$ case. Suppose that $\mathcal{A} = \{A_n : n < \omega\} \subseteq E^{[\infty]}$ is almost disjoint.

Step 1. Choose any $x_0 \in \langle A_0 \rangle$, so that:

$$\langle x_0 \rangle \cap \langle A_0 \rangle \subseteq \langle x_0 \rangle \,.$$

Step 2. Choose $x_1 \in \langle A_1 \rangle$ large enough, so that:

$$\langle x_0, x_1 \rangle \cap \langle A_0 \rangle \subseteq \langle x_0 \rangle \,,$$
$$\langle x_0, x_1 \rangle \cap \langle A_1 \rangle \subseteq \langle x_0, x_1 \rangle \,.$$

Step 3. Choose $x_2 \in \langle A_2 \rangle$ large enough, so that:

$$\langle x_0, x_1, x_2 \rangle \cap \langle A_0 \rangle \subseteq \langle x_0 \rangle \,,$$
$$\langle x_0, x_1, x_2 \rangle \cap \langle A_1 \rangle \subseteq \langle x_0, x_1 \rangle \,,$$
$$\langle x_0, x_1, x_2 \rangle \cap \langle A_2 \rangle \subseteq \langle x_0, x_1, x_2 \rangle \,.$$

and so on. Then $(x_n)_{n<\omega}$ is almost disjoint from $\mathcal{A}$.

Analytic mad families

Consider equipping E with the discrete topology, and $E^{\mathbb{N}}$ with the product topology. Since E is countable, $E^{\mathbb{N}}$ is Polish. Then $E^{[\infty]} \subseteq E^{\mathbb{N}}$ is a closed subspace, so the subspace topology of $E^{[\infty]}$ is also Polish.

Problem (Smythe, 2019)

Is there no analytic mad family $\mathcal{A} \subseteq E^{[\infty]}$ of block subspaces?

Current status. This is open, but Smythe has a partial positive answer.

Proof for $[\omega]^\omega$ case.

1. Given $\mathcal{X} \subseteq [\omega]^\omega$, and $\mathcal{H} \subseteq [\omega]^\omega$ coideal, define "$\mathcal{X}$ is $\mathcal{H}$-Ramsey".
2. Show that if $\mathcal{X} \subseteq [\omega]^\omega$ analytic, and $\mathcal{H} \subseteq [\omega]^\omega$ selective coideal, then $\mathcal{X}$ is $\mathcal{H}$-Ramsey.
3. Given $\mathcal{A}$ almost disjoint, we define $\mathcal{H}(\mathcal{A})$. Show that $\mathcal{H}(\mathcal{A})$ is selective coideal.
4. Show that if $\overline{\mathcal{A}}$ is $\mathcal{H}(\mathcal{A})$-Ramsey, $\mathcal{A}$ is not maximal.

Proof for $E^{[\infty]}$ case.

1. Given $\mathcal{X} \subseteq E^{[\infty]}$, and $\mathcal{H} \subseteq E^{[\infty]}$ "coideal", define "$\mathcal{X}$ is $\mathcal{H}$-strategically Ramsey".
2. Show that if $\mathcal{X} \subseteq E^{[\infty]}$ analytic, and $\mathcal{H} \subseteq E^{[\infty]}$ "selective coideal", then $\mathcal{X}$ is $\mathcal{H}$-strategically Ramsey.
3. Given $\mathcal{A}$ almost disjoint, we define $\mathcal{H}(\mathcal{A})$. Assume that $\mathcal{H}(\mathcal{A})$ is a "selective coideal".
4. Show that if $\overline{\mathcal{A}}$ is $\mathcal{H}(\mathcal{A})$-strategically Ramsey, $\mathcal{A}$ is not maximal.

Step 1 - Define "$\mathcal{H}$-strategically Ramsey". We let capital letters $A, B, C, \dots \in E^{[\infty]}$ denote infinite block sequences, and small letters $a, b, c, \dots \in E^{[<\infty]}$ denote finite block sequences.

Definition (Gowers game)

The *Gowers game* played below $[a, A]$, denoted as $G[a, A]$, is the following game:

I	$A_0 \leq A$		$A_1 \leq A$		$\cdots$
II		$x_0 \in \langle A_0 \rangle$		$x_1 \in \langle A_1 \rangle$	$\cdots$

The outcome of this game is the sequence $a^\frown(x_k)_{k<\omega} \in E^{[\infty]}$.

If $A = (x_0, x_1, \dots)$ is a block sequence, we let $A/n := (x_n, x_{n+1}, \dots)$.

Definition (Asymptotic game)

The *asymptotic game* played below $[a, A]$, denoted as $F[a, A]$, is the following game:

I	A/n_0		A/n_1		$\cdots$
II		$x_0 \in \langle A/n_0 \rangle$		$x_1 \in \langle A/n_1 \rangle$	$\cdots$

The outcome of this game is the sequence $a^\frown (x_k)_{k<\omega} \in E^{[\infty]}$.

Definition

A subset $\mathcal{H} \subseteq E^{[\infty]}$ is a *semicoideal* if it satisfies the following properties:

1. (Cofinite) If $A \in \mathcal{H}$, then $A/n \in \mathcal{H}$ for all n.
2. (Upward-closed) If $A \in \mathcal{H}$ and $A \leq B$, then $B \in \mathcal{H}$.

Definition

A subset $\mathcal{X} \subseteq E^{[\infty]}$ is *$\mathcal{H}$-strategically Ramsey* if for all $A \in \mathcal{H}$ and $a \in E^{[<\infty]}$, there exists some $B \leq A$ where $B \in \mathcal{H}$ such that one of the following holds:

1. **I** has a strategy in $F[a, B]$ to reach $\mathcal{X}^c$.
2. **II** has a strategy in $G[a, B]$ to reach $\mathcal{X}$.

Step 2 - Analytic sets are $\mathcal{H}$-strategically Ramsey.

Theorem (Smythe, 2018)

If $\mathcal{X} \subseteq E^{[\infty]}$ is analytic, and $\mathcal{H} \subseteq E^{[\infty]}$ is a full "selective" semicoideal, then $\mathcal{X}$ is $\mathcal{H}$-strategically Ramsey.

Step 3 - Define $\mathcal{H}(\mathcal{A})$. Let $\mathcal{A} \subseteq E^{[\infty]}$ be an almost disjoint family. We define:

$$\mathcal{H}(\mathcal{A}) := \left\{ B \in E^{[\infty]} : \exists^{\infty} A \in \mathcal{A} \text{ s.t. } \dim(\langle A \rangle \cap \langle B \rangle) = \infty \right\}.$$

Fact

$\mathcal{H}(\mathcal{A})$ is a "selective" semicoideal.

What about fullness? Is $\mathcal{H}(\mathcal{A})$ a full semicoideal?

Definition

A mad family $\mathcal{A} \subseteq E^{[\infty]}$ is *full* if $\mathcal{H}(\mathcal{A})$ is full.

Theorem (Smythe, 2019)

If $\mathfrak{a}_{\mathrm{vec},\mathbb{F}} = \mathfrak{c}$, then there exists a full mad family.

Problem (Smythe, 2019)

1. (ZFC) Is there a full mad family?
2. (ZFC) Is every mad family full?

Step 4 - Show that if $\mathcal{A}$ is maximal, then $\overline{\mathcal{A}}$ is not $\mathcal{H}(\mathcal{A})$-strategically Ramsey. If $\mathcal{A}$ is an almost disjoint family, we define:

$$\overline{\mathcal{A}} := \{B \in E^{[\infty]} : B \leq A \text{ for some } A \in \mathcal{A}\}.$$

Note that:

- $\mathcal{A} \subseteq \overline{\mathcal{A}}$.
- $\mathcal{H}(\mathcal{A}) \cap \overline{\mathcal{A}} = \emptyset$.
- If $\mathcal{A}$ is analytic, so is $\overline{\mathcal{A}}$.

Proposition

Let $\mathcal{A} \subseteq E^{[\infty]}$ be a mad family. Then for any $B \in \mathcal{H}(\mathcal{A})$,

1. **II** *has a strategy in $F[B]$ to reach $\overline{\mathcal{A}}$, and*
2. **I** *has a strategy in $G[B]$ to reach $\mathcal{H}(\mathcal{A})$ (and hence $\overline{\mathcal{A}}^c$).*

General approach

The key proposition to proving the consistency of $\mathfrak{a} < \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$ is the following:

Theorem (Smythe, 2019 + Brendle-García Ávila, 2017)

$\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$.

Since $\mathfrak{a} < \mathrm{non}(\mathcal{M})$ in the random model, $\mathfrak{a} < \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$ is consistent.

The following theorem is the main stepping stone.

Theorem (Brendle-García Ávila, 2017)

non$(\mathcal{M}) \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}_2}$, *where* $\mathbb{F}_2$ *is the field of two elements.*

Smythe showed that this is enough to show that non$(\mathcal{M}) \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$.

Sketch of proof.

Define the map $s : E(\mathbb{F}) \to E(\mathbb{F}_2)$ by:

$$s\left(\lambda_{n_0} e_{n_0} + \cdots + \lambda_{n_k} e_{n_k}\right) := e_{n_0} + \cdots + e_{n_k},$$

i.e. s replaces all non-zero coefficients of e_n with 1. Let $\mathcal{A} \subseteq E^{[\infty]}(\mathbb{F})$ be an almost disjoint family of size less than $\mathrm{non}(\mathcal{M})$.

$\mathcal{A} \subseteq E^{[\infty]}(\mathbb{F})$

$\downarrow$

$s[\mathcal{A}] \subseteq E^{[\infty]}(\mathbb{F}_2) \xrightarrow{\text{B-G, 2017}} B' \in E^{[\infty]}(\mathbb{F}_2)$ a.d. from $s[\mathcal{A}]$

$\uparrow$ s surjective

$B \in E^{[\infty]}(\mathbb{F})$, $s(B) = B'$ a.d. from $\mathcal{A}$

$\square$

A characterisation non($\mathcal{M}$)

We present a characterisation of the cardinal non($\mathcal{M}$) used in the proof of Brendle-García Ávila.

Definition

Let $h : \omega \to \omega$ be a function such that $\lim_{n\to\infty} h(n) = \infty$. The cardinal $\mathfrak{b}_h(p \neq^*)$ is defined by:

$$\mathfrak{b}_h(p \neq^*) := \min \left\{ |\mathcal{F}| : \begin{array}{c} \mathcal{F} \subseteq \omega^\omega \text{ and } \forall \text{partial } g : \omega \to \omega \text{ s.t.} \\ |\operatorname{dom}(g)| = \infty \text{ and } g \leq h, \\ \text{there is some } f \in \mathcal{F} \text{ s.t.} \\ \exists^\infty n \in \operatorname{dom}(g)\ f(n) = g(n) \end{array} \right\}.$$

Lemma

For any $h, h' : \omega \to \omega$ *with* $\lim_{n\to\infty} h(n) = \lim_{n\to\infty} h'(n) = \infty$, $\mathfrak{b}_h(p \neq^*) = \mathfrak{b}_{h'}(p \neq^*)$.

Thus, we may let $\mathfrak{b}(pbd \neq^*)$ be the cardinal $\mathfrak{b}_h(p \neq^*)$ for any such h.

Proposition

$\mathrm{non}(\mathcal{M}) = \max\{\mathfrak{b}, \mathfrak{b}(pbd \neq^*)\}$.

Recall that we proved the following "diagonalisation" lemma for block subspaces.

Lemma

Let $A \in E^{[\infty]}$, and let $x_0, \ldots, x_n$ be non-zero vectors. Then there exists some M such that for any $x \notin \langle A \rangle$ such that whenever $x > M$ (i.e. $\min(\mathrm{supp}(x)) > M$),

$$\langle x_0, \ldots, x_n, x \rangle \cap \langle A \rangle = \langle x_0, \ldots, x_n \rangle \cap \langle A \rangle .$$

Using this lemma, and by mimicking the proof of $\mathfrak{b} \leq \mathfrak{a}$, Smythe proved that:

Proposition (Smythe, 2019)

$\mathfrak{b} \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$.

Proof of $\mathfrak{b}(pbd \neq^*) \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}_2}$

We're only left with showing that $\mathfrak{b}(pbd \neq^*) \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}_2}$. We may fix some arbitrary $h \in \omega^\omega$ such that $\lim_{n\to\infty} h(n) = \infty$, and show that $\mathfrak{b}_{h+1}(p \neq^*) \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}_2}$.

Let $\mathcal{A} \subseteq E^{[\infty]}$ be an almost disjoint family such that $|\mathcal{A}| < \mathfrak{b}_{h+1}(p \neq^*)$. The proof outline is as follows:

1. Given a partial function g from ω to ω, define a block sequence B^g.
2. Conversely, for any block sequence $A \in \mathcal{A}$, define a (total) function $f_A : \omega \to \omega$.
3. Since $\{f_A : A \in \mathcal{A}\}$ is of size $< \mathfrak{b}_{h+1}(p \neq^*)$, there is a partial function g, with $|\mathrm{dom}(g)| = \infty$ and $g \leq h + 1$, such that for all $A \in \mathcal{A}$, $g(n) \neq f_A(n)$ for all but finitely many $n \in \mathrm{dom}(g)$.
4. Show that if $g(n) \neq f_A(n)$ for all but finitely many $n \in \mathrm{dom}(g)$, then B^g and A are almost disjoint.

Step 1 - Define a block sequence B^g given a partial function $g : \omega \to \omega$. Fix some $A_0 \in \mathcal{A}$, and fix any block sequence A_1 so that $\langle A_0 \rangle \cap \langle A_1 \rangle = \{0\}$. We choose vectors c_n^i, d_n^i so that:

1. c_n^i, d_n^i are defined for $i \leq h(n)$.
2. $c_n^i \in \langle A_0 \rangle$ for all n, i.
3. $d_n^i \in \langle A_1 \rangle$ for all n, i.
4. $c_n^i < c_n^{i+1}$.
5. $d_n^i < d_n^{i+1}$.
6. $d_{n-1}^{h(n-1)+1} < c_n^i < d_n^i < c_{n+1}^0$ for $i \leq h(n) + 1$,

We also define:

1. $c_n := \sum_{i \leq h(n)} c_n^i$.
2. $d_n := \sum_{i \leq h(n)} d_n^i$.
3. $b_n^k := c_n + d_n - c_n^k - d_n^k$.

If $g : \omega \to \omega$ is a partial function with $|\mathrm{dom}(g)| = \infty$ and $g \leq h + 1$, we define:

$$B^g := (b_n^{g(n)-1})_{n \in \mathrm{dom}(g) \wedge g(n) > 0}.$$

Step 2 - Given a block sequence A, define a (total) function $f_A : \omega \to \omega$.

- Given two vectors x, y, we say that x is *interval inside* y if $y = z + x + w$ for some vectors z, w such that $z < x < w$.
- If A is a block sequence, we say that x is *compatible* with A if x is interval inside some $y \in \langle A \rangle$.

Claim

If $k \neq k'$ and $b_n^k, b_n^{k'}$ are both compatible with A, then $c_n^k, c_n^{k'}, d_n^k, d_n^{k'} \in \langle A \rangle$.

Claim

For any $A \in \mathcal{A}$ and almost all n, there are at most one k such that b_n^k is compatible with A.

Thus, given $A \in \mathcal{A}$ we shall define:

$$f_A(n) := \begin{cases} k+1, & \text{if only } b_n^k \text{ is compatible with } A, \\ 0, & \text{if none of the } b_n^k\text{'s are compatible with } A. \end{cases}$$

Step 3 and 4 - Show that B^g is almost disjoint from all of $A \in \mathcal{A}$. Since $|\mathcal{A}| < \mathfrak{b}_{h+1}(p \neq^*)$, let $g : \omega \to \omega$ be a partial function so that for all $A \in \mathcal{A} \cup \{n \mapsto 0\}$, $g(n) \neq f_A(n)$ for almost all $n \in \mathrm{dom}(g)$.

Claim

B^g is almost disjoint from every $A \in \mathcal{A}$.

Therefore, $\mathcal{A}$ is not mad, completing the proof.

Summary

1. Is every mad family of block subspaces uncountable?
   - Yes - using a special diagonalisation lemma proved by studying the supports of vectors.
2. Are there no analytic mad families of block subspaces?
   - Still open. There are no analytic full mad families of block subspaces - proved using the theory of $\mathcal{H}$-strategically Ramsey sets.
3. Relationship between $\mathfrak{a}$ and $\mathfrak{a}_{\text{vec},\mathbb{F}}$?
   - $\mathfrak{a} < \mathfrak{a}_{\text{vec},\mathbb{F}}$ is consistent. $\mathfrak{a} > \mathfrak{a}_{\text{vec},\mathbb{F}}$ is open.
   - It follows from the ZFC inequality $\text{non}(\mathcal{M}) \leq \mathfrak{a}_{\text{vec},\mathbb{F}}$, and that $\mathfrak{a} < \text{non}(\mathcal{M})$ in the random model.