Projections between Ramsey quasi-orders and topological Ramsey spaces (Part 1)

Clement Yung
Joint work with Spencer Unger

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Welcome!

Today's agenda:

1. Discuss a few examples of Ramsey-theoretic statements.
2. Introduce Ramsey quasi-orders, structures that carry analytic versions of these Ramsey-theoretic statements.
3. Study projection maps that transfer these Ramsey-theoretic statements.
4. Prove the (non-)existence of such maps between examples of topological Ramsey spaces (Ellentuck space, Hindman space, space of strong subtrees).

The most basic form of pigeonhole principle:

Fact

For all $\mathcal{O} \subseteq \mathbb{N}$ and $X \in [\mathbb{N}]^\infty$, there exists some $Y \in [X]^\infty$ such that $Y \subseteq \mathcal{O}$ or $Y \subseteq \mathcal{O}^c$.

Statement. In the quasi-order $(\mathbb{N}^\infty, \subseteq)$, given:

- (Structure) $X \in \mathbb{N}^\infty$, and
- (Partition) $\mathcal{O} \subseteq \mathbb{N}$,

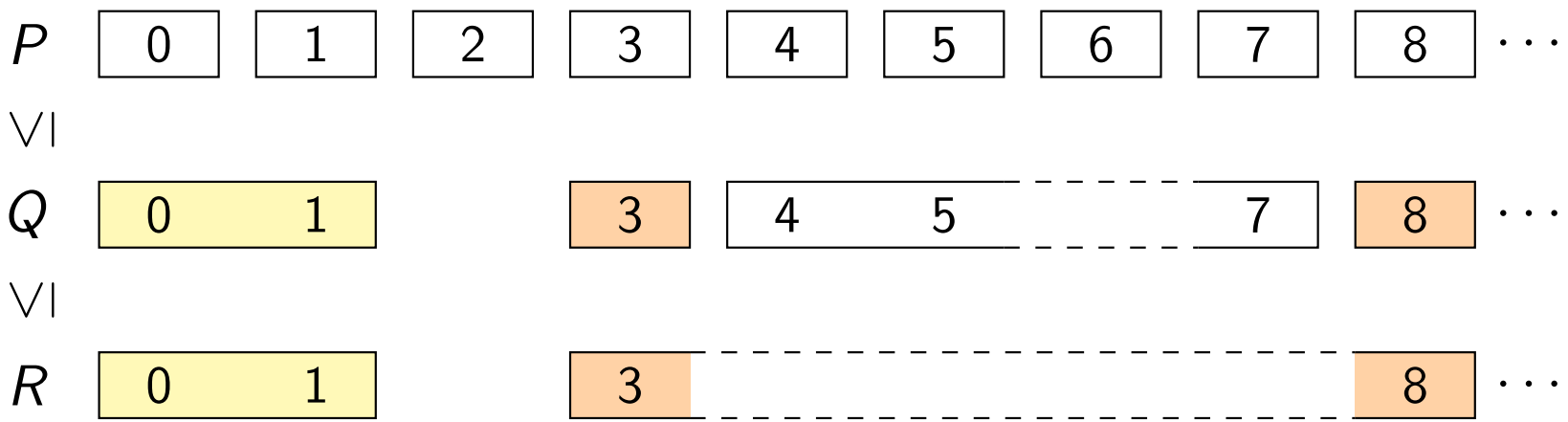
there are:

- (Substructure) $Y \in [X]^\infty$, such that
- (Decides $\mathcal{O}$) $Y \subseteq \mathcal{O}$ or $Y \subseteq \mathcal{O}^c$.

Hindman's theorem

Hindman's theorem is a partition theorem about *block sequences* of **FIN** (non-empty finite sets of $\mathbb{N}$).

Block sequences $(\mathbf{FIN}^{[\infty]}, \leq)$



Notation. Given $P = (x_m)_{m<\omega}$ and $Q = (y_n)_{n<\omega}$ in $\mathbf{FIN}^{[\infty]}$, write:

$$Q \leq P \iff \text{Every } y_n \text{ is a finite union of } x_m\text{'s.}$$

Notation. If $Q = (x_n)_{n<\omega} \in \mathbf{FIN}^{[\infty]}$, then let:

$$\langle Q \rangle := \{\text{Non-empty finite union of } x_n\text{'s}\}.$$

Theorem (Hindman, 1974)

For all $\mathcal{O} \subseteq \mathbf{FIN}$ and $P \in \mathbf{FIN}^{[\infty]}$, there exists some $Q \leq P$ such that $\langle Q \rangle \subseteq \mathcal{O}$ or $\langle Q \rangle \subseteq \mathcal{O}^c$.

Statement. In the quasi-order $(\mathbf{FIN}^{[\infty]}, \leq)$, given:

- (Structure) $P \in \mathbf{FIN}^{[\infty]}$, and
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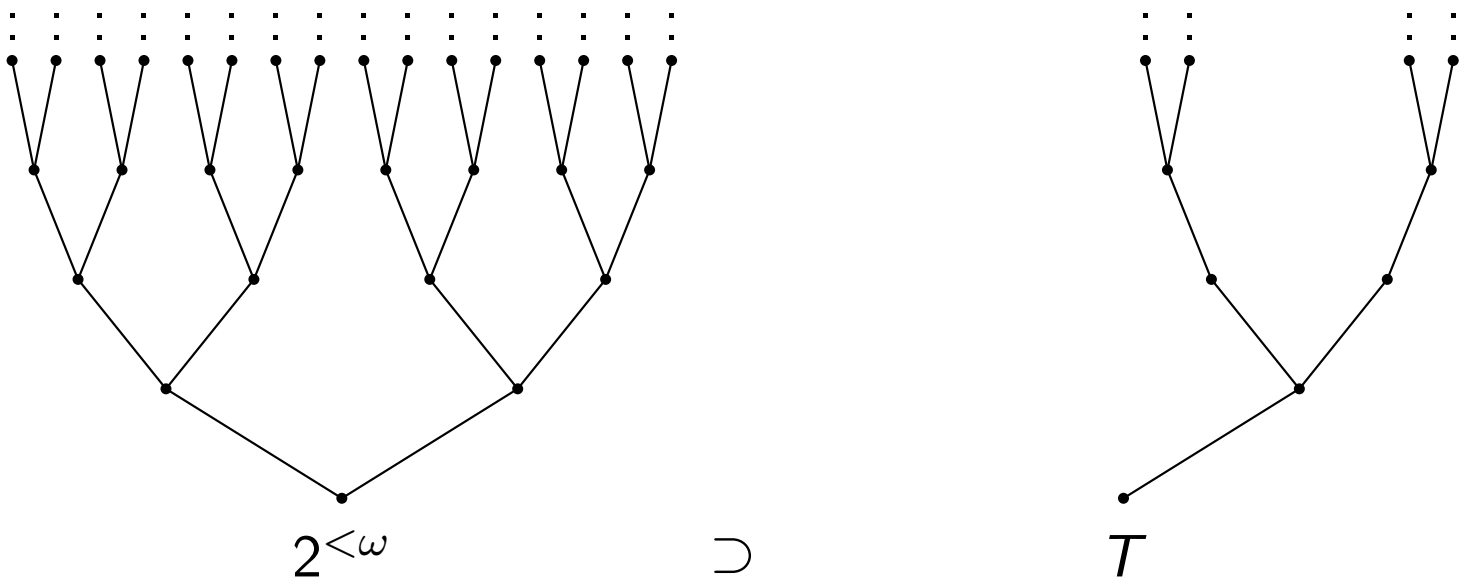
there are:

- (Substructure) $Q \leq P$, such that
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Halpern–Läuchli theorem (Strong binary trees)

(A version of) the Halpern–Läuchli theorem involves partitions of nodes in an (infinite) strong binary tree.

Strong (binary) subtrees $(\mathcal{S}_\infty, \subseteq)$



We identify each tree with the set of splitting nodes.

Theorem (Halpern–Läuchli)

For all $\mathcal{O} \subseteq 2^{<\omega}$ and $T \in \mathcal{S}_\infty$, there exists some strong subtree $U \subseteq T$ such that $U \subseteq \mathcal{O}$ or $U \subseteq \mathcal{O}^c$.

Statement. In the quasi-order $(\mathcal{S}_\infty, \subseteq)$, given:

- (Structure) $T \in \mathcal{S}_\infty$, and
- (Partition) $\mathcal{O} \subseteq 2^{<\omega}$,

there are:

- (Substructure) a strong subtree $U \subseteq T$, such that
- (Decides $\mathcal{O}$) $U \subseteq \mathcal{O}$ or $U \subseteq \mathcal{O}^c$.

Ramsey-theoretic statements

Hindman theorem

Halpern–Läuchli theorem

Pigeonhole principle

Which Ramsey-theoretic statement can be derived from which?

(Topological??) Ramsey-theoretic statements

Hindman theorem

Halpern–Läuchli theorem

Pigeonhole principle

~~**Which Ramsey-theoretic statement can be derived from which?**~~

Not quite the question we're trying to answer today!

Analytic sets are Ramsey

A set of sequences can be equipped with the *first-difference metric*:

$$d((x_n)_{n<\omega}, (y_n)_{n<\omega}) = \min\left\{\frac{1}{2^n} : x_n \neq y_n\right\}.$$

Ellentuck space $([\mathbb{N}]^\infty, \subseteq)$



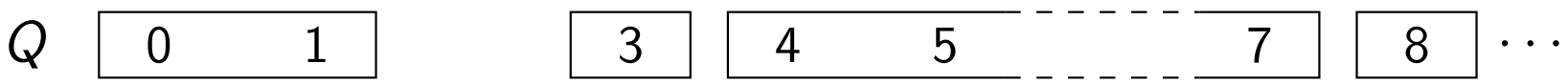
$$X = \{0, 1, 3, 8, \dots\}$$

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Block sequences $(\mathbf{FIN}^{[\infty]}, \leq)$



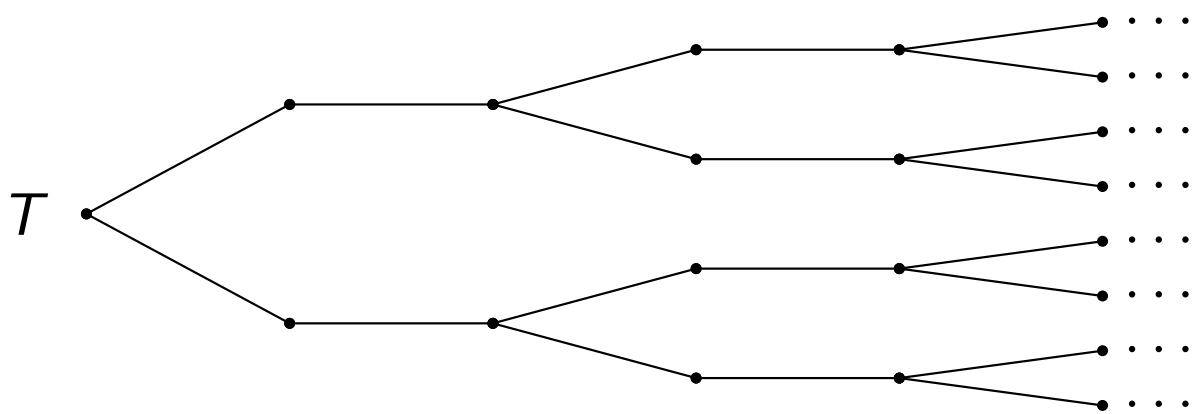
$$Q = \{\{0,1\},\{3\},\{4,5,7\},\{8\},\dots\}$$

Analytic sets are Ramsey

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Strong subtrees $(\mathcal{S}_\infty, \subseteq)$



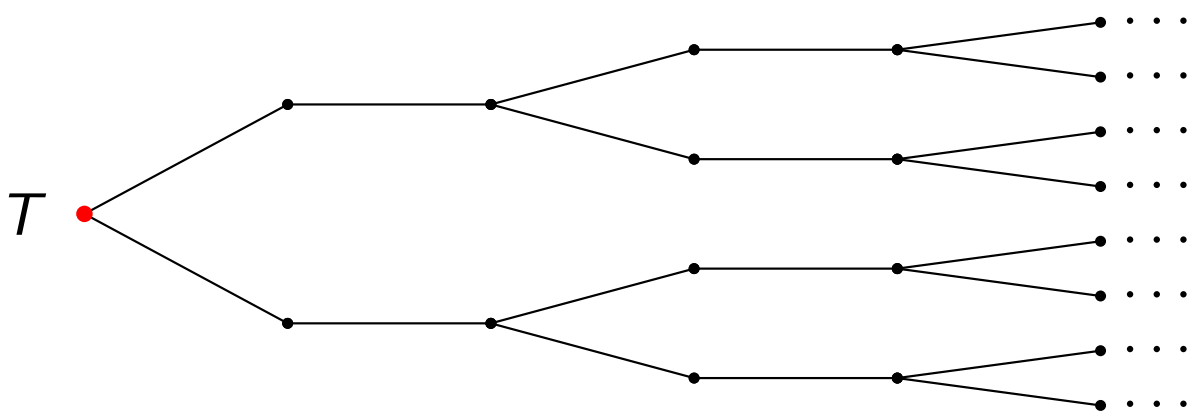
$$T = \{\emptyset, (0,0), (1,1), (0,0,0,0), (0,0,1,1), (1,1,0,0), (1,1,1,1), \dots\}$$

Analytic sets are Ramsey

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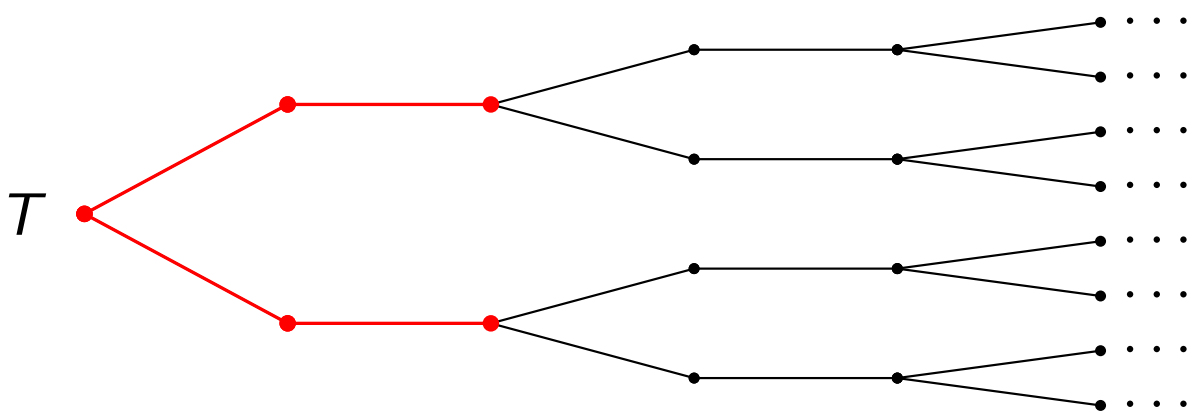
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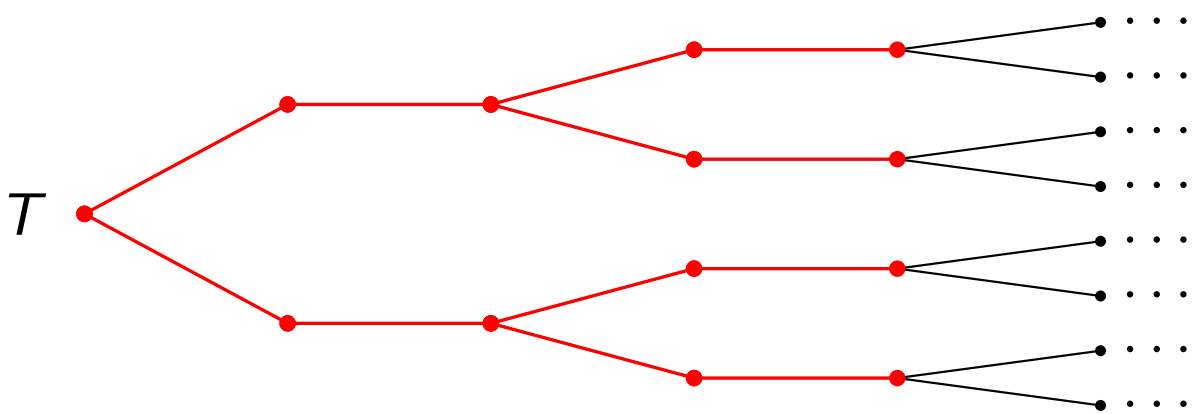
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Strong subtrees $(\mathcal{S}_\infty, \subseteq)$



$$T = \{\emptyset, (0,0), (1,1), (0,0,0,0), (0,0,1,1), (1,1,0,0), (1,1,1,1), \dots\}$$

Under this metric, $[\mathbb{N}]^\infty$, $\mathbf{FIN}^{[\infty]}$ and $\mathcal{S}_\infty$ are Polish spaces, so we can talk about Borel, analytic etc. subsets of these spaces.

Notation (for $[\mathbb{N}]^\infty$)**.** Let $[Y] = \{Z \in [\mathbb{N}]^\infty : Z \subseteq Y\}$.

Theorem (Silver, 1970)

Every analytic subset $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is Ramsey — that is, for all $X \in [\mathbb{N}]^\infty$, there exists some infinite $Y \subseteq X$ such that $[Y] \subseteq \mathcal{X}$ or $[Y] \subseteq \mathcal{X}^c$.

Under this metric, $[\mathbb{N}]^\infty$, $\mathbf{FIN}^{[\infty]}$ and $\mathcal{S}_\infty$ are Polish spaces, so we can talk about Borel, analytic etc. subsets of these spaces.

Notation (for $\mathbf{FIN}^{[\infty]}$)**.** Let $[Q] = \{R \in \mathbf{FIN}^{[\infty]} : R \leq Q\}$.

Theorem (Milliken, 1975)

Every analytic subset $\mathcal{X} \subseteq \mathbf{FIN}^{[\infty]}$ is Ramsey — that is, for all $P \in \mathbf{FIN}^{[\infty]}$, there exists some $Q \leq P$ such that $[Q] \subseteq \mathcal{X}$ or $[Q] \subseteq \mathcal{X}^c$.

Under this metric, $[\mathbb{N}]^\infty$, $\mathbf{FIN}^{[\infty]}$ and $\mathcal{S}_\infty$ are Polish spaces, so we can talk about Borel, analytic etc. subsets of these spaces.

Notation (for $\mathcal{S}_\infty$)**.** Let $[S] = \{S \in \mathcal{S}_\infty : S \subseteq U\}$.

Theorem (Milliken, 1981)

Every analytic subset $\mathcal{X} \subseteq \mathcal{S}_\infty$ is Ramsey — that is, for all $T \in \mathcal{S}_\infty$, there exists some strong subtree $U \subseteq T$ such that $[U] \subseteq \mathcal{X}$ or $[U] \subseteq \mathcal{X}^c$.

Topological Ramsey-theoretic statements

Milliken theorem #1
(Analytic Hindman)

Milliken theorem #2
(Analytic Halpern–Läuchli)

Silver's theorem

Which topological Ramsey-theoretic statement can be derived from which?

Ramsey quasi-orders and quasi-open maps

Ellentuck space	**Block sequences**	**Strong subtrees**
Polish space $[\mathbb{N}]^\infty$	Polish space $\mathbf{FIN}^{[\infty]}$	Polish space $\mathcal{S}_\infty$
Analytic $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is Ramsey	Analytic $\mathcal{X} \subseteq \mathbf{FIN}^{[\infty]}$ is Ramsey	Analytic $\mathcal{X} \subseteq \mathcal{S}_\infty$ is Ramsey

Notation. Let $[B] := \{C \in \mathcal{R} : C \leq B\}$.

Definition

A *Ramsey quasi-order* is a quasi-order $(\mathcal{R}, \leq)$ such that:

1. $(\mathcal{R}, \leq)$ is an *analytic quasi-order*, i.e.:
   - 1.1 $\mathcal{R}$ is a Polish space.
   - 1.2 $\leq \subseteq \mathcal{R} \times \mathcal{R}$ is analytic.
2. (Analytic Ramsey property) If $\mathcal{X} \subseteq \mathcal{R}$ analytic and $A \in \mathcal{R}$, there exists some $B \leq A$ such that $[B] \subseteq \mathcal{X}$ or $[B] \subseteq \mathcal{X}^c$.

Ramsey quasi-order $(\mathcal{R}, \leq_{\mathcal{R}})$ $\xrightarrow[f : \mathcal{R} \to \mathcal{S}]{}$ Analytic quasi-order $(\mathcal{S}, \leq_{\mathcal{S}})$

To transfer the analytic Ramsey property, we need f to be:

Continuous, Cofinal, Monotone, Quasi-open

$f^{-1}(\mathcal{X})$ analytic $\xleftarrow{\text{Pullback via } f}$ $\mathcal{X} \subseteq \mathcal{S}$ analytic

$A \in \mathcal{R}$, $f(A) \leq X$ $\qquad$ $X \in \mathcal{S}$

$\downarrow$ $(\mathcal{R}, \leq_{\mathcal{R}})$ Ramsey $\qquad$ $\vdots$

$[B] \subseteq f^{-1}(\mathcal{X})$ or $[B] \subseteq f^{-1}(\mathcal{X}^c)$ $\xrightarrow{\text{Quasi-open}}$ $[Y] \subseteq f``[B] \subseteq \mathcal{X}$ or $[Y] \subseteq f``[B] \subseteq \mathcal{X}^c$

$B \leq A$ $\qquad$ $Y \leq f(B) \leq f(A) \leq X$

Definition

A map $f : \mathcal{R} \to \mathcal{S}$ is *quasi-open* if for all $A \in \mathcal{R}$, there exists some $X \in \mathcal{S}$ such that $[X] \subseteq f\text{“}[A]$.

Lemma

Let $(\mathcal{R}, \leq_{\mathcal{S}})$ be a Ramsey quasi-order, and let $(\mathcal{S}, \leq_{\mathcal{S}})$ be an analytic quasi-order. If there is a continuous cofinal monotone quasi-open $f : \mathcal{R} \to \mathcal{S}$, then $(\mathcal{S}, \leq_{\mathcal{S}})$ is also a Ramsey quasi-order.

But why (monotone) quasi-open maps?

1. Motivated by other frameworks on transferring Ramsey properties.
2. **Closely related to maps transferring forcing-theoretic properties.**

If $\mathcal{E} \subseteq \mathcal{R}$, we let:

$$\mathcal{E}^{\uparrow} := \{A \in \mathcal{R} : B \leq A \text{ for some } B \in \mathcal{E}\}.$$

Lemma

Let $(\mathcal{R}, \leq_{\mathcal{S}})$ and $(\mathcal{S}, \leq_{\mathcal{S}})$ be quasi-orders, with a monotone quasi-open map $f : \mathcal{R} \to \mathcal{S}$. If G is a $(\mathcal{R}, \leq_{\mathcal{R}})$-generic filter over a model M, then $f``(G)^{\uparrow}$ is a $(\mathcal{S}, \leq_{\mathcal{S}})$-generic filter over M.

Definition

A map $f : \mathcal{R} \to \mathcal{S}$ is *Alaoglu–Birkhoff convergent* (or just *AB-convergent*) if for all dense open $\mathcal{D} \subseteq \mathcal{S}$, $f^{-1}(\mathcal{D})$ contains a dense open subset of $\mathcal{R}$.

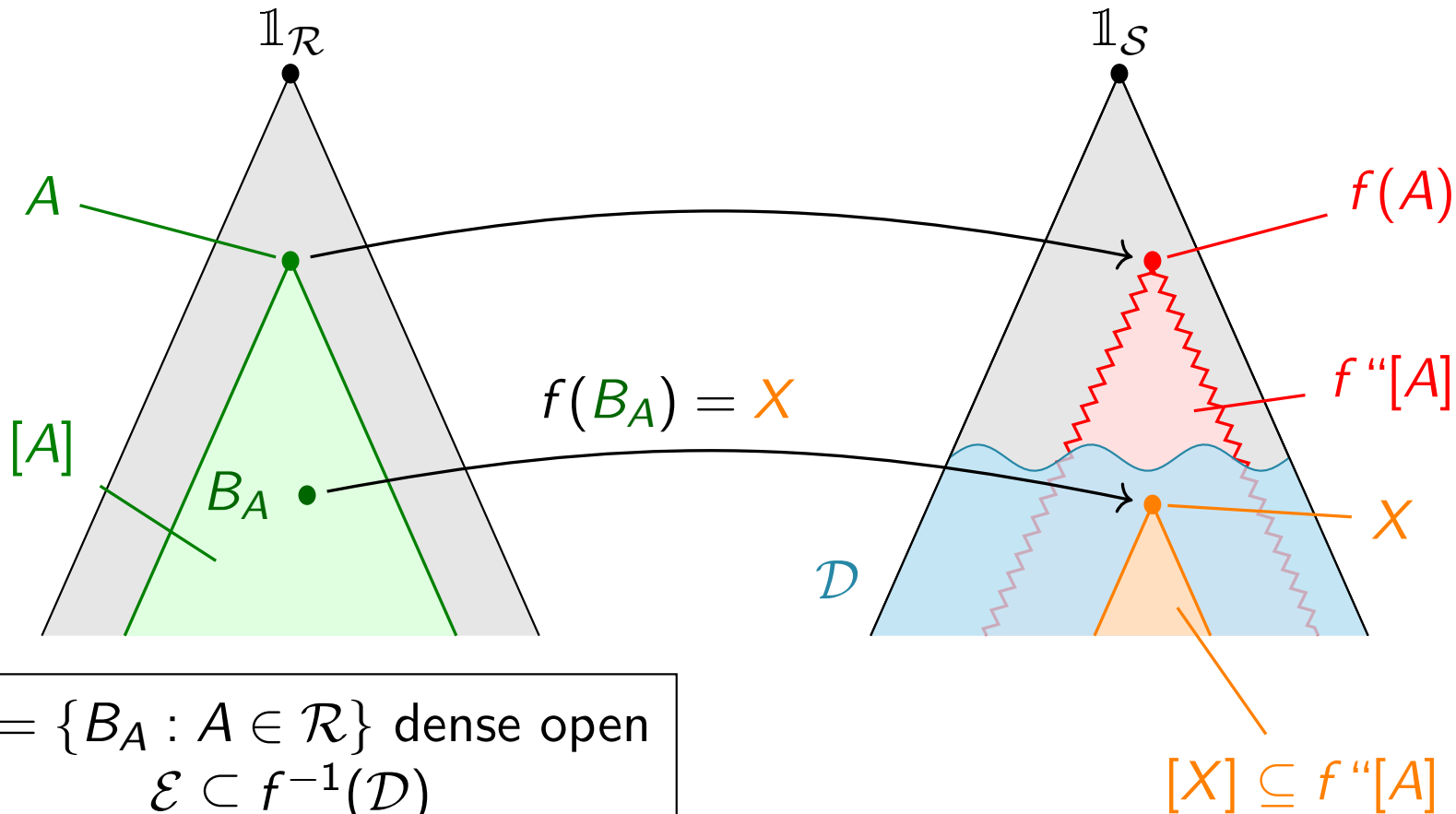
Lemma

Let $f : \mathcal{R} \to \mathcal{S}$ be a monotone map. The following are equivalent:

1. *f is Alaoglu–Birkhoff convergent.*
2. *If G is a $(\mathcal{R}, \leq_{\mathcal{R}})$-generic filter over a model M, then $f``(G)^{\uparrow}$ is a $(\mathcal{S}, \leq_{\mathcal{S}})$-generic filter over M.*

Lemma

Every monotone quasi-open map is Alaoglu–Birkhoff convergent.



$\mathcal{E} := \{B_A : A \in \mathcal{R}\}$ dense open
$\mathcal{E} \subseteq f^{-1}(\mathcal{D})$

Therefore:

$(\mathcal{R}, \leq_{\mathcal{R}})$ **is Ramsey-theoretically stronger than** $(\mathcal{S}, \leq_{\mathcal{S}})$

$\Downarrow$

$(\mathcal{R}, \leq_{\mathcal{R}})$ **is forcing-theoretically stronger than** $(\mathcal{S}, \leq_{\mathcal{S}})$

What about the converse?

Lemma

Let $(\mathcal{R}, \leq_{\mathcal{R}})$ be an analytic quasi-order, and let $(\mathcal{S}, \leq_{\mathcal{S}})$ be a Ramsey quasi-order. If $f : \mathcal{R} \to \mathcal{S}$ is continuous monotone Alaoglu–Birkhoff convergent, then f is quasi-open.

Proof.

Since $f``[A]$ is analytic (as $\leq$ is analytic and f is continuous), the set:

$$\mathcal{D} := \{X \in \mathcal{S} : [X] \cap f``[A] = \emptyset \text{ or } [X] \subseteq f``[A]\}$$

is dense open. This implies $f^{-1}(\mathcal{D})$ is contains a dense open subset of $\mathcal{R}$, so there exists some $B \leq A$ such that $[f(B)] \subseteq \mathcal{D}$. Since $[f(B)] \cap f``[A] \neq \emptyset$, we have that $[f(B)] \subseteq f``[A]$. □

Lemma

Let $(\mathcal{R}, \leq_{\mathcal{R}})$ be an analytic quasi-order, and let $(\mathcal{S}, \leq_{\mathcal{S}})$ be a Ramsey quasi-order. If $f : \mathcal{R} \to \mathcal{S}$ is continuous monotone Alaoglu–Birkhoff convergent, then f is quasi-open.

If, however, $(\mathcal{R}, \leq_{\mathcal{R}})$ and $(\mathcal{S}, \leq_{\mathcal{S}})$ are each one of $([\mathbb{N}]^{\infty}, \subseteq)$, $(\mathbf{FIN}^{[\infty]}, \leq)$ or $(\mathcal{S}_{\infty}, \subseteq)$, then we do **not** need to assume that f is continuous. Thus, for these three Ramsey quasi-orders:

$(\mathcal{R}, \leq_{\mathcal{R}})$ **is Ramsey-theoretically stronger than** $(\mathcal{S}, \leq_{\mathcal{S}})$

$\Updownarrow$

$(\mathcal{R}, \leq_{\mathcal{R}})$ **is forcing-theoretically stronger than** $(\mathcal{S}, \leq_{\mathcal{S}})$

Definition

A *weak projection* is a monotone Alaoglu–Birkhoff convergent map.

Ramsey quasi-orders

Block sequences $(\mathbf{FIN}^{[\infty]}, \leq)$ Strong subtrees $(\mathcal{S}_\infty, \subseteq)$

Ellentuck space $([\mathbb{N}]^\infty, \subseteq)$

Question of the day: **Which weakly projects into which?**

Easiest cases

Ramsey quasi-orders

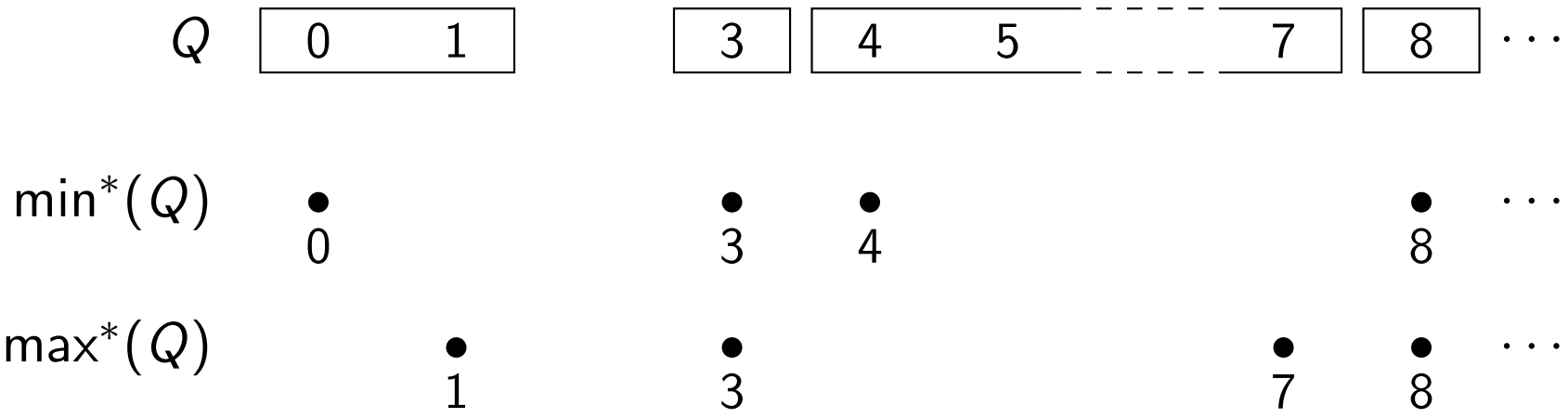
Block sequences $(\mathbf{FIN}^{[\infty]}, \leq)$

Strong subtrees $(\mathcal{S}_\infty, \subseteq)$



Ellentuck space $([\mathbb{N}]^\infty, \subseteq)$

Projection $(\mathbf{FIN}^{[\infty]}, \leq) \to ([\mathbb{N}]^\infty, \subseteq)$



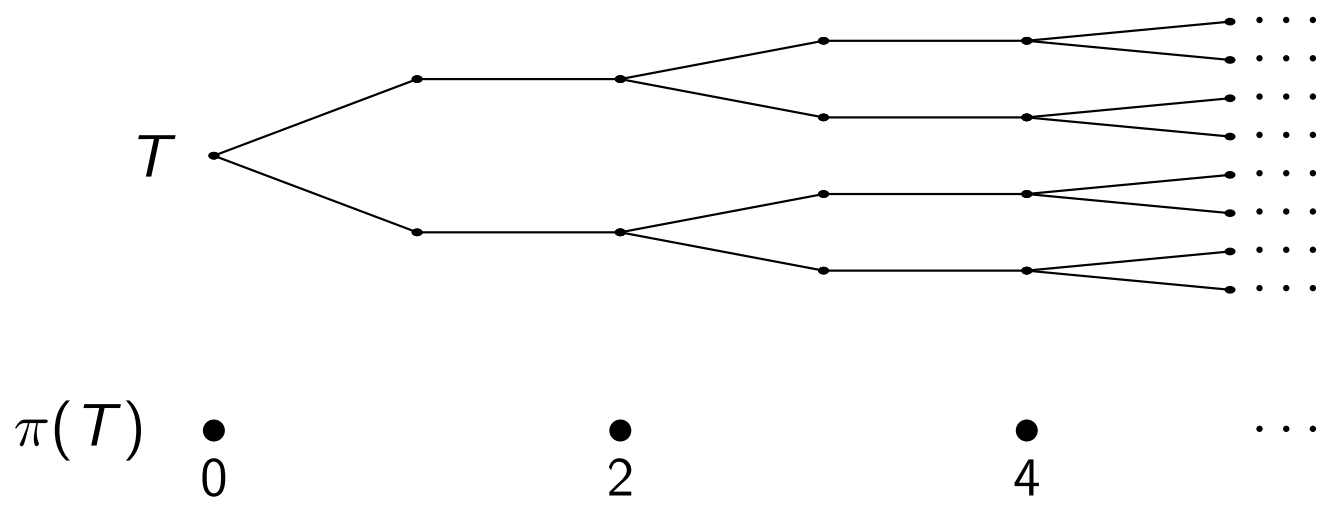
Lemma

The maps $\min^*, \max^* : \mathbf{FIN}^{[\infty]} \to [\mathbb{N}]^\infty$ *defined by:*

$$\min^*((p_n)_{n<\omega}) = \{\min(p_n) : n < \omega\},$$
$$\max^*((p_n)_{n<\omega}) = \{\max(p_n) : n < \omega\}.$$

are cofinal continuous monotone, and $f``[P] = [f(P)]$ *for all* P.

Projection $(\mathcal{S}_\infty, \subseteq) \to ([\mathbb{N}]^\infty, \subseteq)$



Lemma

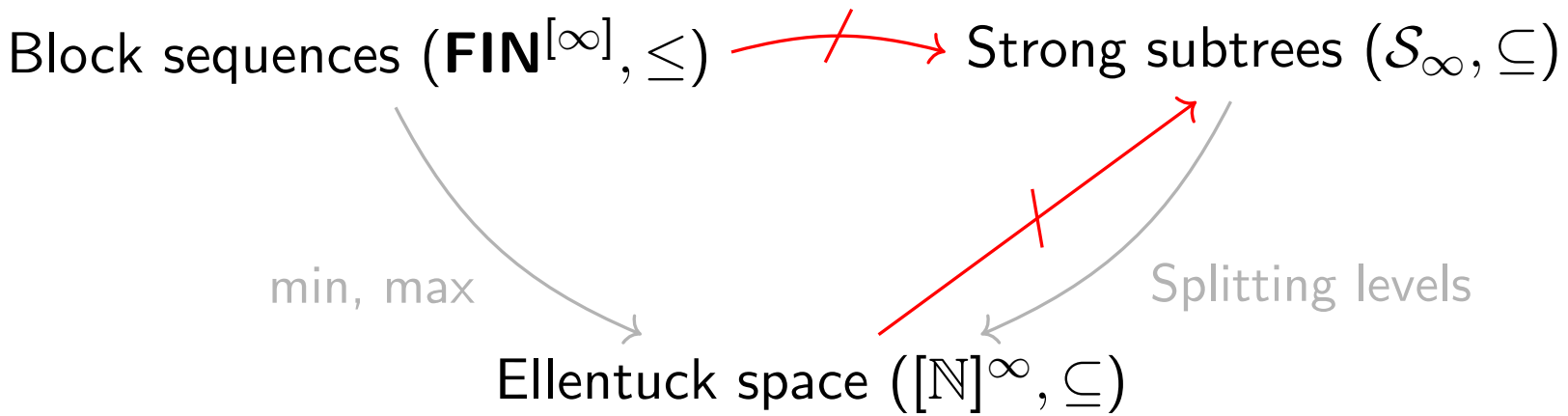
The map $\pi : \mathcal{S}_\infty \to [\mathbb{N}]^\infty$ *defined by:*

$$\pi(T) := \{\mathrm{dom}(s) : s \in T \text{ is a splitting node}\}$$

is cofinal continuous monotone, and $\pi``[T] = [\pi(T)]$ *for all* T.

Still pretty easy

Ramsey quasi-orders



No weak projection $([\mathbb{N}]^\infty, \subseteq) \to (\mathcal{S}_\infty, \subseteq)$

Write $Y \subseteq^* X$ iff $X \setminus Y$ is finite. Recall that:

- $([\mathbb{N}]^\infty, \subseteq)$ densely embeds into $([\mathbb{N}]^\infty, \subseteq^*)$.
- $([\mathbb{N}]^\infty, \subseteq^*)$ is σ-closed (hence does not add reals).

If $f : [\mathbb{N}]^\infty \to \mathcal{S}_\infty$ is a weak projection, and G is $([\mathbb{N}]^\infty, \subseteq)$-generic, then $H = f\text{“}(G)^\uparrow$ is $(\mathcal{S}_\infty, \subseteq)$-generic, and:

$$M \subseteq M[H] \subseteq M[G].$$

But forcing with $(\mathcal{S}_\infty, \subseteq)$ adds a real, so:

$$(\omega^\omega)^M \subsetneq (\omega^\omega)^{M[H]} \subseteq (\omega^\omega)^{M[G]} = (\omega^\omega)^M. \qquad \lightning$$

No weak projection $(\mathbf{FIN}^{[\infty]}, \leq) \to (\mathcal{S}_\infty, \subseteq)$

We use the same idea, by defining some relation $\leq^*$ such that:

- $(\mathbf{FIN}^{[\infty]}, \leq)$ densely embeds into $(\mathbf{FIN}^{[\infty]}, \leq^*)$.
- $(\mathbf{FIN}^{[\infty]}, \leq^*)$ is σ-closed (hence does not add reals).

$$B \leq^* A \iff \exists n\, B/n \leq A \quad (B/n := B \text{ w/o first } n \text{ blocks})$$



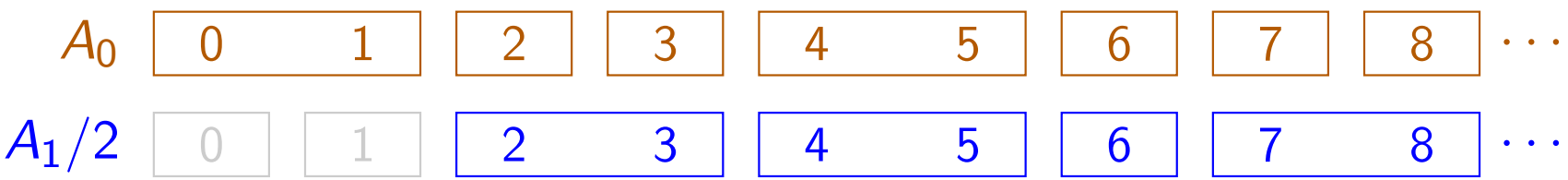
$$A_1 \not\leq A_0$$

No weak projection $(\mathbf{FIN}^{[\infty]}, \leq) \to (\mathcal{S}_\infty, \subseteq)$

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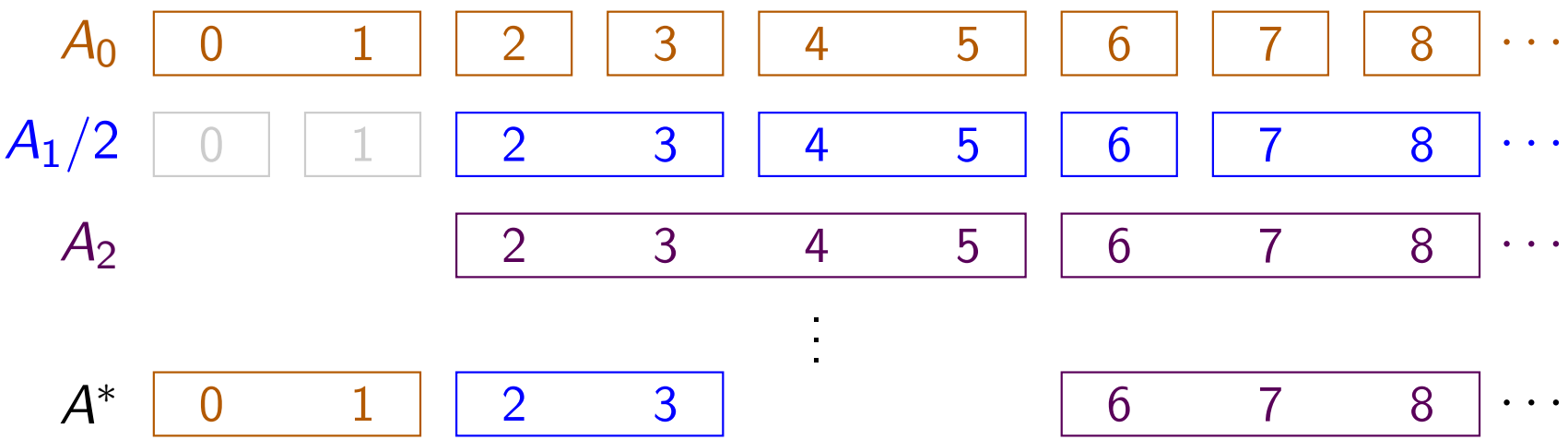
$$A_1/2 \leq A_0 \implies A_1 \leq^* A_0$$

No weak projection $(\mathbf{FIN}^{[\infty]}, \leq) \to (\mathcal{S}_\infty, \subseteq)$

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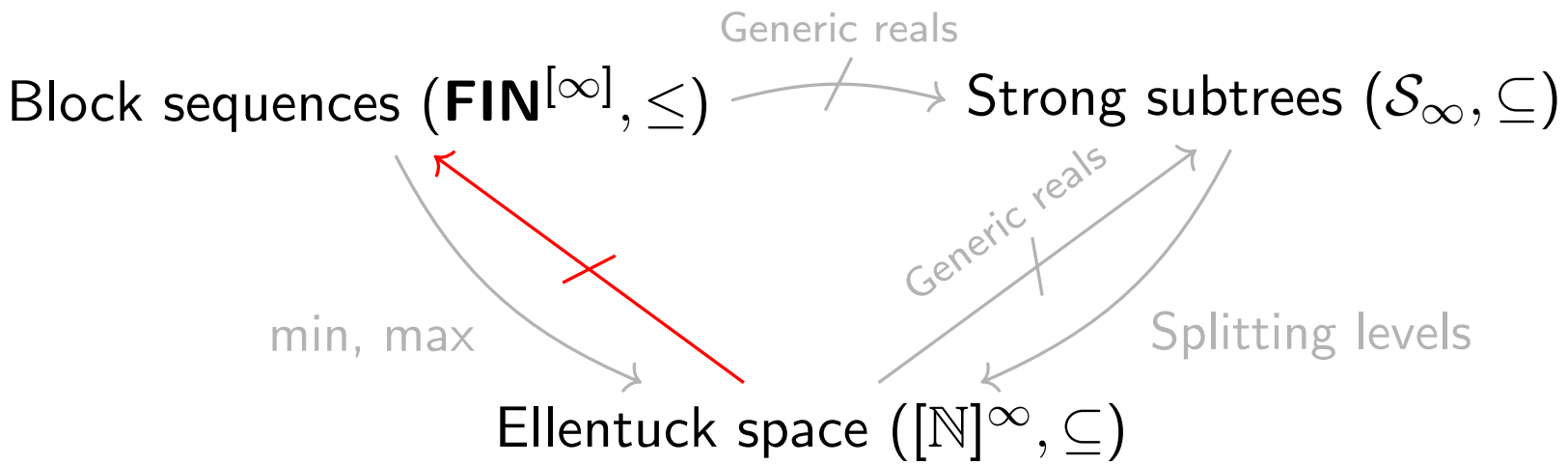
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$$B \leq^* A \iff \exists n\, B/n \leq A \quad (B/n := B \text{ w/o first } n \text{ blocks})$$



Not trivial, but not difficult

Ramsey quasi-orders



We use the **Tukey theory of ultrafilters**.

Definition

Let $\mathcal{U}, \mathcal{V}$ be directed sets. $\mathcal{V}$ is *Tukey below* $\mathcal{U}$, denoted $\mathcal{V} \leq_T \mathcal{U}$, if there is a cofinal function $f : \mathcal{U} \to \mathcal{V}$.

Write $\mathcal{V} \equiv_T \mathcal{U}$ if $\mathcal{V} \leq_T \mathcal{U}$ and $\mathcal{U} \leq_T \mathcal{V}$.

A *Tukey class* below some $\mathcal{U}$ is an equivalence class of ultrafilters Tukey below $\mathcal{U}$.

If $\mathcal{U}$ has some special properties, what can we say about the Tukey classes below $\mathcal{U}$?

Fact 1.1. $([\mathbb{N}]^\infty, \subseteq)$-generic filters are selective.

Fact

If G is $([\mathbb{N}]^\infty, \subseteq)$-generic over M, then:

$$M[G] \models G \text{ is a selective ultrafilter.}$$

Fact 1.2. Selective ultrafilters are Tukey minimal.

Theorem (Raghavan–Todorčević, 2012)

If G is a selective ultrafilter, and $\mathcal{V}$ is an ultrafilter such that $\mathcal{V} \leq_T G$, then either $\mathcal{V}$ is principal or $\mathcal{V} \equiv_T G$.

Fact 2.1. $(\mathbf{FIN}^{[\infty]}, \leq)$-generic filters are *stable ordered-union ultrafilters*.

Lemma (Blass, 1987)

If H is $(\mathbf{FIN}^{[\infty]}, \leq)$-generic over M, then there is some ultrafilter $\mathcal{U}_{\mathbf{FIN}} \equiv_T H$ such that:

$$M[H] \models \mathcal{U}_{\mathbf{FIN}} \text{ is a stable ordered-union ultrafilter.}$$

Fact 2.2. Tukey class classification of $\mathcal{U}_{\mathbf{FIN}}$.

Theorem (Dobrinen–Todorčević, 2011)

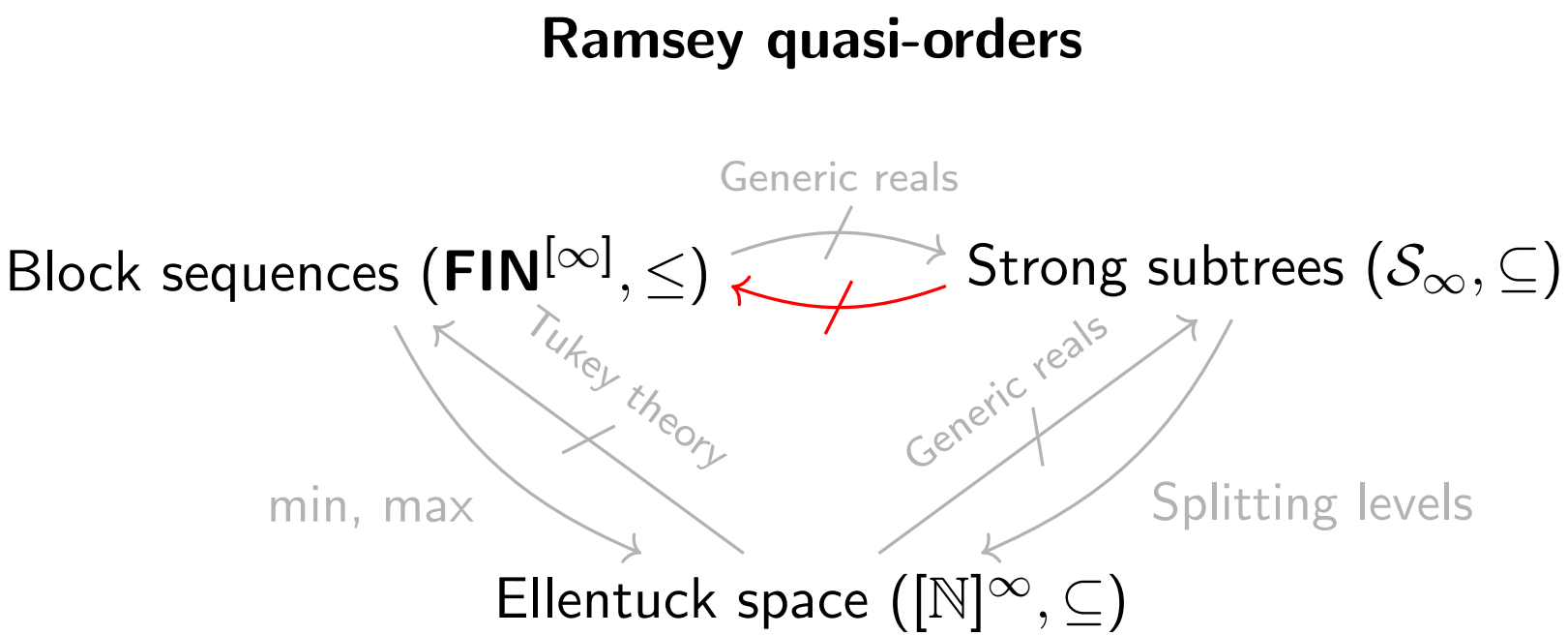
If $\mathcal{U}_{\mathbf{FIN}}$ is a stable ordered-union ultrafilter, then there is some non-principal ultrafilter $\mathcal{V}$ such that $\mathcal{V} <_T \mathcal{U}_{\mathbf{FIN}}$.

No weak projection $([\mathbb{N}]^\infty, \subseteq) \to (\mathbf{FIN}^{[\infty]}, \leq)$

Otherwise:

1. Let G be $([\mathbb{N}]^\infty, \subseteq)$-generic over M, which is a selective ultrafilter.
2. $H := f``(G)^\uparrow$ is $(\mathbf{FIN}^{[\infty]}, \leq)$-generic, and $H \leq_T G$ (witnessed by $f \restriction G$). Thus, $\mathcal{U}_{\mathbf{FIN}} \equiv_T H \leq_T G$.
3. $\mathcal{U}_{\mathbf{FIN}}$ is a stable ordered-union ultrafilter; take any non-principal $\mathcal{V} <_T \mathcal{U}_{\mathbf{FIN}}$.
4. Then $\mathcal{V} <_T \mathcal{U}_{\mathbf{FIN}} \leq_T G$, a contradiction.

Oh no


Ramsey quasi-orders
Generic reals
Block sequences $(\mathbf{FIN}^{[\infty]}, \leq)$
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Tukey theory
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> *The world ain't all sunshine and rainbows.*
> — *Rocky Balboa*

Why previous tools won't work:

- **(Generic reals)** Forcing with $([\mathbb{N}]^\infty, \subseteq)$ and $(\mathbf{FIN}^{[\infty]}, \leq)$ do not add reals, but forcing with $(\mathcal{S}_\infty, \subseteq)$ does. Thus, the former can't weakly project into the latter, but obviously the argument fails for the converse.
- **(Tukey theory)** There is no natural notion of ultrafilters for strong subtrees.

Suppose for a contradiction that there is a weak projection $f : \mathcal{S}_\infty \to \mathbf{FIN}^{[\infty]}$. We may assume that f is continuous. This gives us two weak projections $\mathcal{S}_\infty \to [\mathbb{N}]^\infty$:

1. $\pi : \mathcal{S}_\infty \to [\mathbb{N}]^\infty$, by:

$$\pi(T) := \{\mathrm{dom}(s) : s \in T \text{ is a splitting node}\}.$$

2. $h := \max^* \circ f : \mathcal{S}_\infty \to [\mathbb{N}]^\infty$.

Goal. Show that π and h are not too different from each other.

Lemma 1 of 3

There exists some $T_0 \in \mathcal{S}_\infty$ and $\psi : \mathbb{N} \to \mathbb{N}$ such that for all $U \subseteq T_0$:

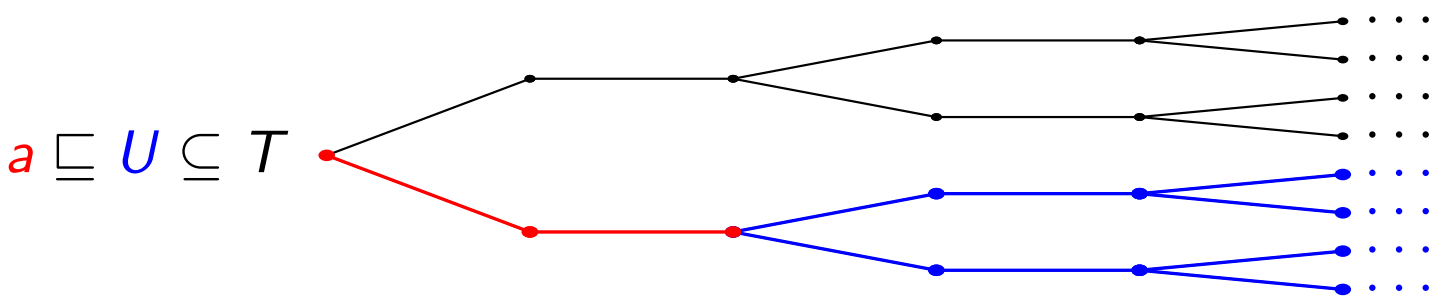
$$\psi``(\pi(U)) \cap h(U) \neq \emptyset.$$

We need some notations. Let $a, b, c, \ldots$ denote *finite* strong subtrees.

Definition

If a is a finite strong subtree of T, then let:

$$[a, T] := \{U \in \mathcal{S}_\infty : U \subseteq T \text{ and } a \sqsubseteq U\}.$$



$U \in [a, T]$

Let $\Phi := \min \circ h$, so $\Phi : \mathcal{S}_\infty \to \mathbb{N}$ is continuous (where $\mathbb{N}$ is discrete). For a strong subtree T and $a \subseteq T$:

- *T accepts a* if Φ is constant on $[a, T]$.
- *T rejects a* if there is no $U \in [a, T]$ that accepts a.
- *T decides a* if T accepts or rejects a.

Claim

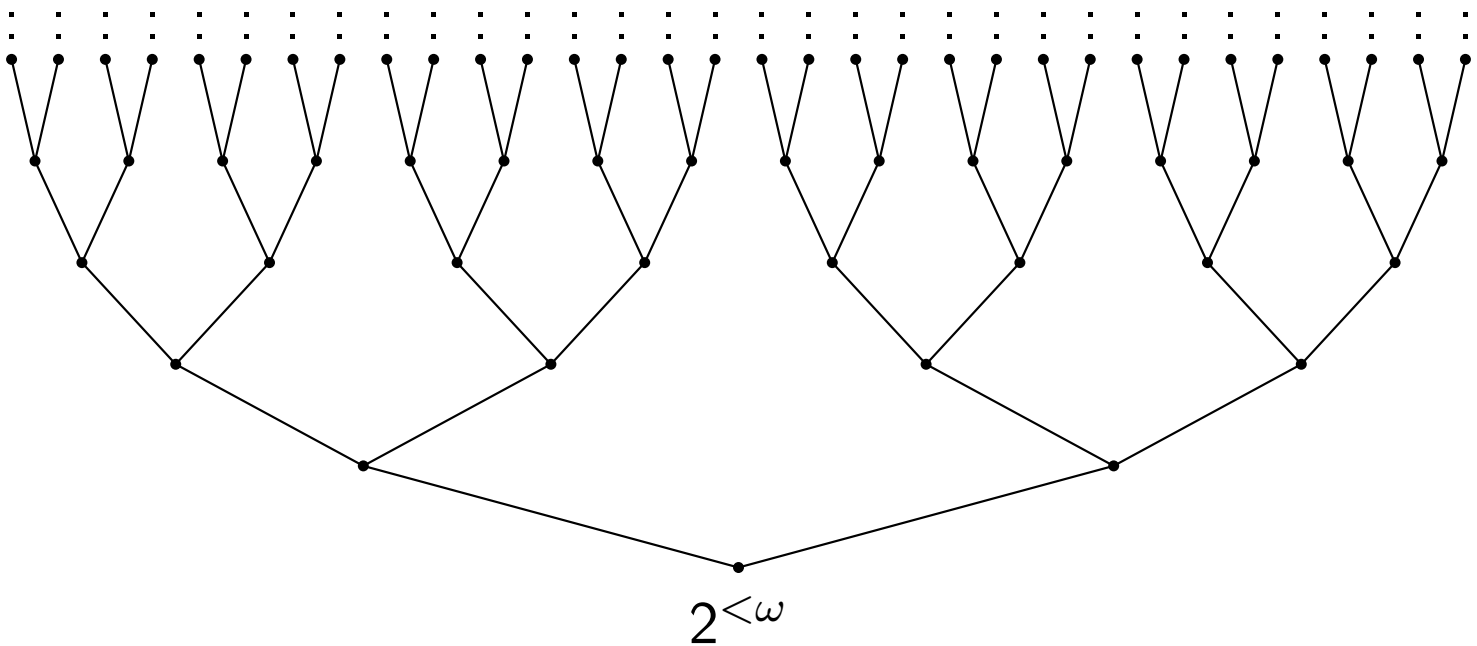
There is some $T \in \mathcal{S}_\infty$ such that:

1. *T decides every $a \subseteq T$. (Classic fusion argument)*
2. *Every $U \subseteq T$ has some $a \sqsubseteq U$ such that T accepts a. (Φ continuous)*
3. *If $a, a' \subseteq T$ are $\sqsubseteq$-minimal elements accepted by T, then:*

$$\Phi{\restriction}[a, T] = \Phi{\restriction}[a', T] \implies \mathrm{height}(a) = \mathrm{height}(a').$$

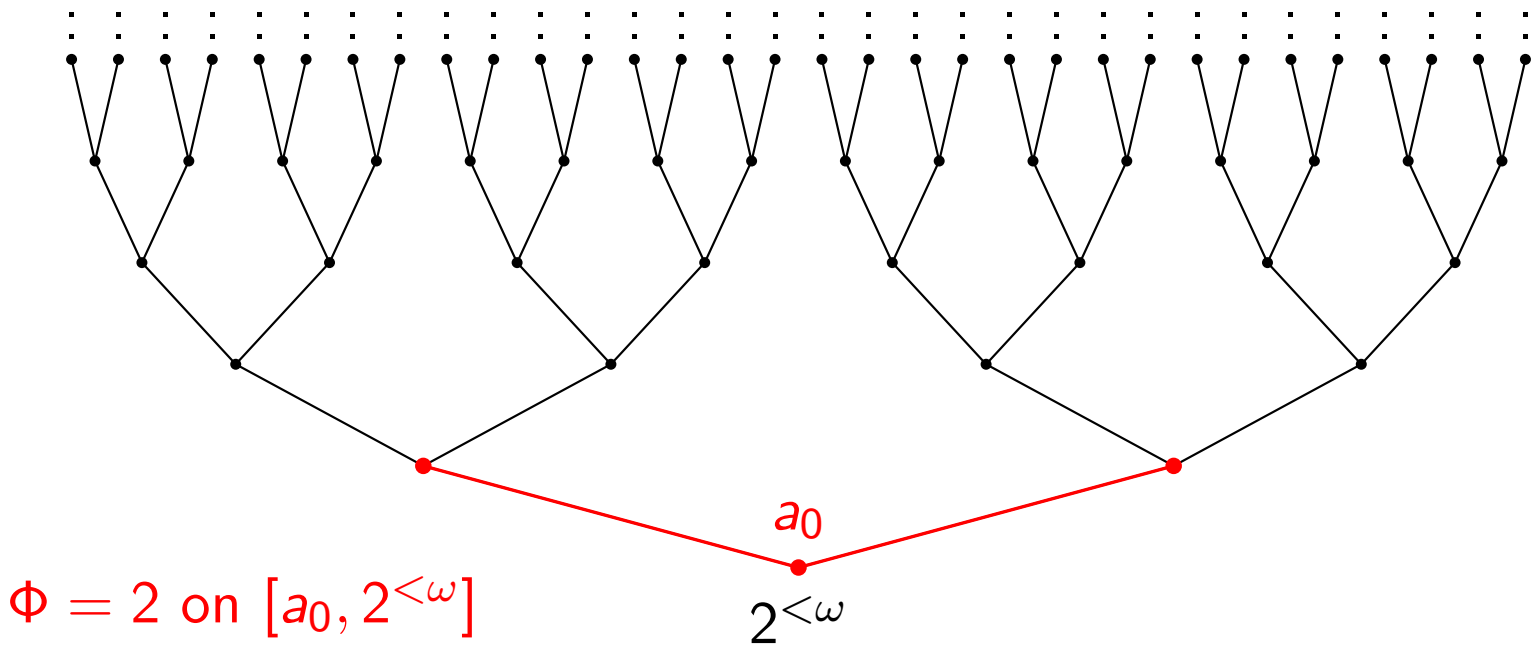
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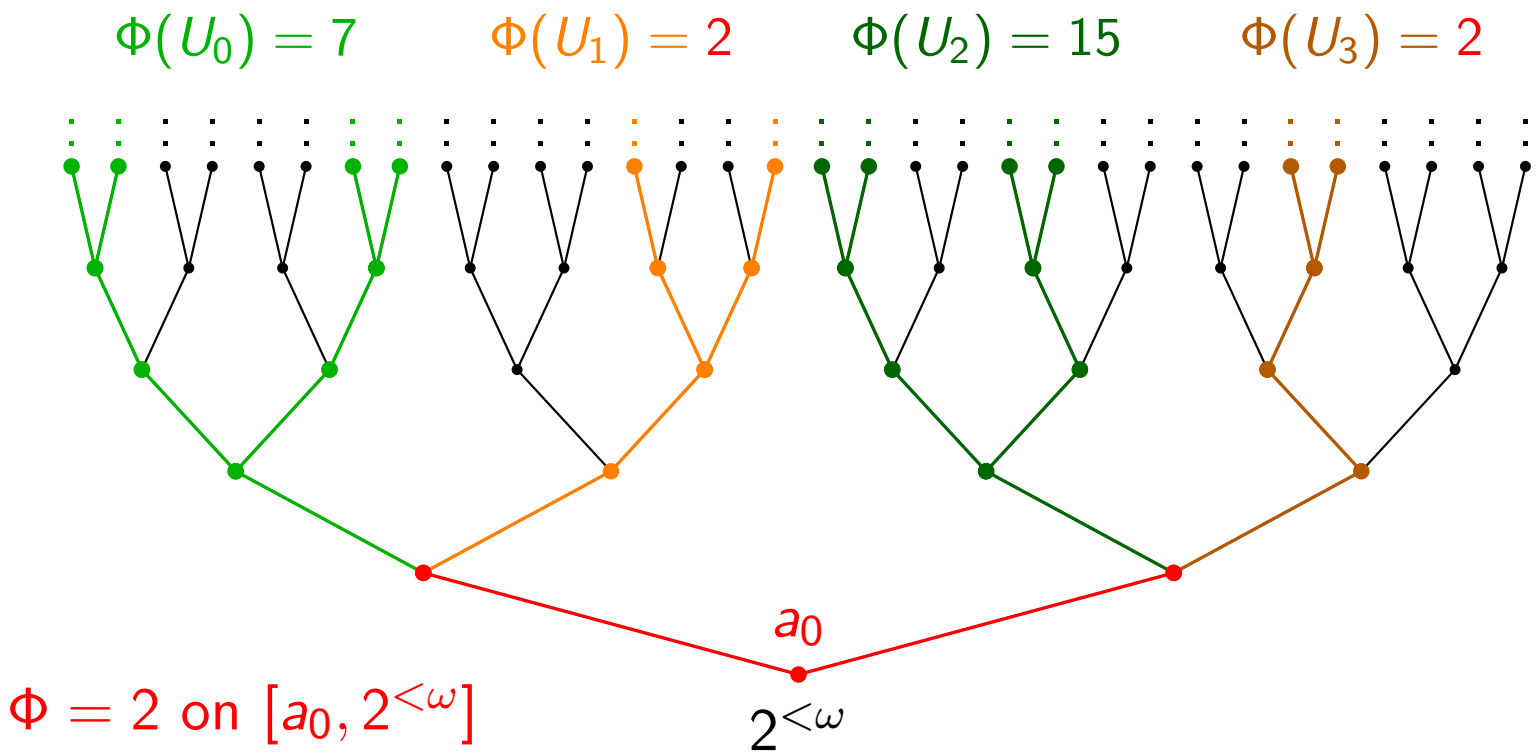
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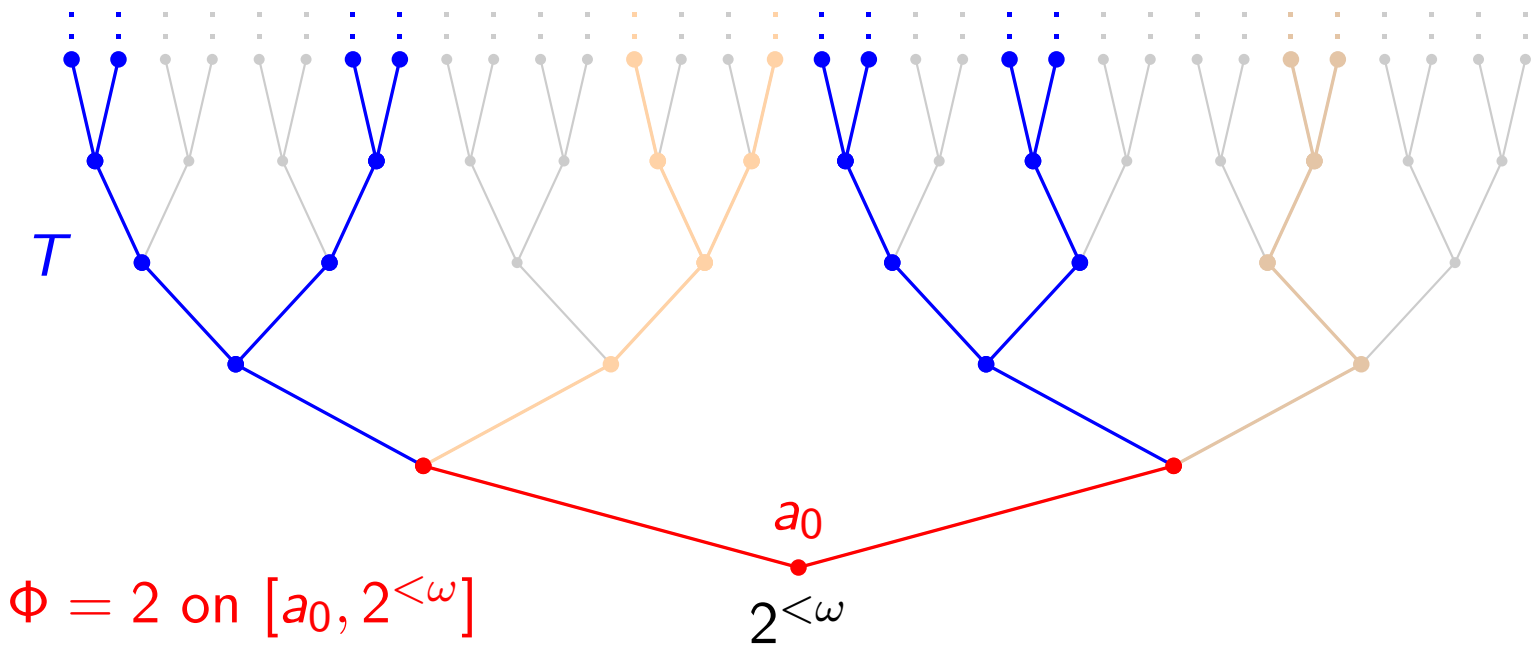
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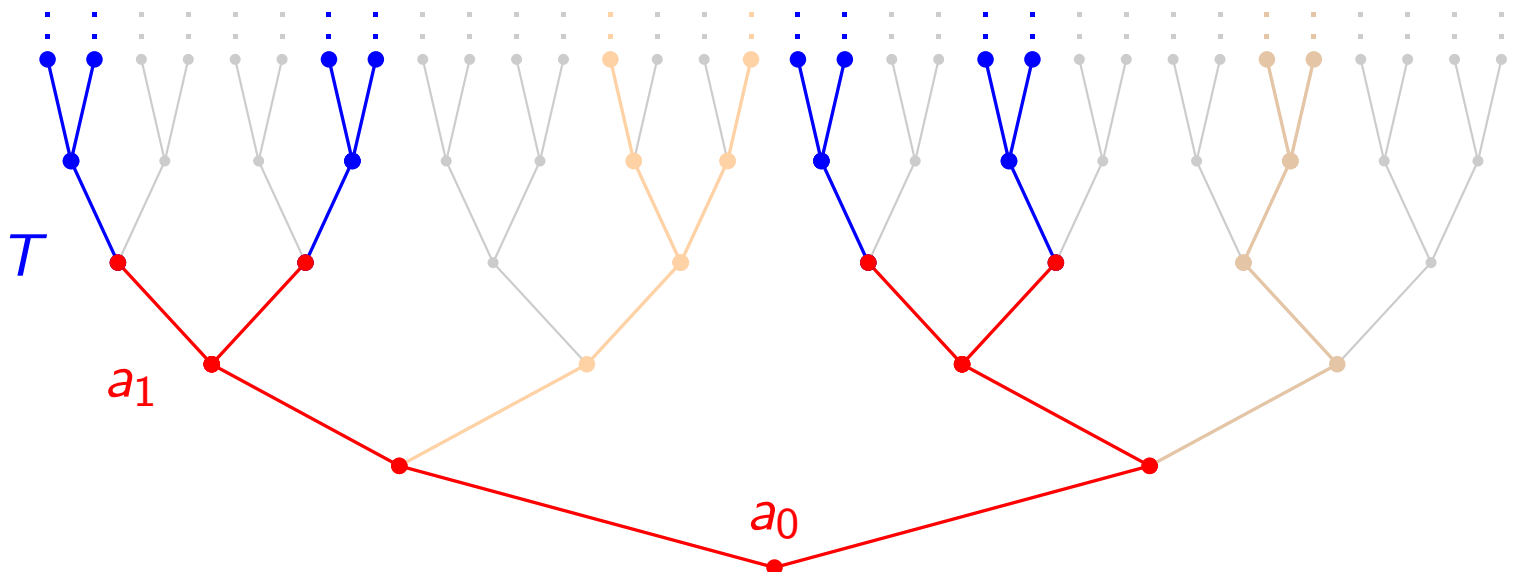
Φ continuous $\implies$ Subtree T that avoids trees U with $\Phi(U) = 2$



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Φ continuous $\implies$ Subtree T that avoids trees U with $\Phi(U) = 2$



We have:

2. *Every $U \subseteq T$ has some $a \sqsubseteq U$ such that T accepts a.*
3. *If $a, a' \subseteq T$ are $\sqsubseteq$-minimal elements accepted by T, then:*

$$\Phi{\restriction}[a, T] = \Phi{\restriction}[a', T] \implies \mathrm{height}(a) = \mathrm{height}(a').$$

For each $U \subseteq T$, let $a_U \sqsubseteq T$ be the shortest initial segment such that T accepts a_U (i.e. Φ is constant on $[a_U, T]$). Then there exists some $g : \mathbb{N} \to \mathbb{N}$ such that for all $U \subseteq T$:

$$\mathrm{height}(a_U) = (g \circ \Phi)(U)$$

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$$\mathrm{height}(a_U) = (g \circ \Phi)(U)$$

We let:

$$\mathcal{D} := \{X \in [\mathbb{N}]^\infty : g{\restriction}X \text{ is constant or injective}\}.$$

$\mathcal{D}$ is a dense open subset of $[\mathbb{N}]^\infty$. Since h is Alaoglu–Birkhoff convergent, there exists some $T_0 \subseteq T$ such that $g{\restriction}h(T_0)$ is injective. Let $\psi : \mathbb{N} \to \mathbb{N}$ such that $\psi(g(n)) = n$ for all n. Then for all $U \subseteq T_0$:

$$\psi``(\pi(U)) \cap h(U) \neq \emptyset.$$ □

Timeout. What we have so far:

- Assume $f : \mathcal{S}_\infty \to \mathbf{FIN}^{[\infty]}$ is a weak projection.
- We have two weak projections:
  - $\pi : \mathcal{S}_\infty \to [\mathbb{N}]^\infty$. (Splitting levels)
  - $h := \max^* \circ f : \mathcal{S}_\infty \to [\mathbb{N}]^\infty$.
- **Lemma 1.** There is some $T_0 \in \mathcal{S}_\infty$ and $\psi : \mathbb{N} \to \mathbb{N}$ such that for all $U \subseteq T_0$:

$$\psi``(\pi(U)) \cap h(U) \neq \emptyset.$$

Let G be $(\mathcal{S}_\infty, \subseteq)$-generic such that $T_0 \in G$.

$$\begin{aligned} \mathcal{V} &:= f``(G)^\uparrow, && (\mathbf{FIN}^{[\infty]}, \leq)\text{-generic} \\ \mathcal{U} &:= \pi``(G)^\uparrow, && ([\mathbb{N}]^\infty, \subseteq)\text{-generic} \\ \mathcal{U}' &:= h``(G)^\uparrow = (\max^*)``(\mathcal{V})^\uparrow. && ([\mathbb{N}]^\infty, \subseteq)\text{-generic} \end{aligned}$$

Let G be $(\mathcal{S}_\infty, \subseteq)$-generic such that $T_0 \in G$.

$$\mathcal{V} := f``(G)^\uparrow, \quad \mathcal{U} := \pi``(G)^\uparrow.$$

Lemma 2 of 3

$\mathcal{V} \in M[\mathcal{U}]$.

This follows from the "almost minimality" of $(\mathcal{S}_\infty, \subseteq)$:

Theorem (Unger-Y., 2026)

If $X \in M[G] \setminus M[\mathcal{U}]$ is a set of ordinals, then $M[X] = M[G]$.

For all $U \subseteq T_0$:

$$\psi``(\pi(U)) \cap h(U) \neq \emptyset.$$

Let G be $(\mathcal{S}_\infty, \subseteq)$-generic such that $T_0 \in G$.

$$\mathcal{U} := \pi``(G)^\uparrow, \quad \mathcal{U}' := h``(G)^\uparrow.$$

Lemma 3 of 3

$M[\mathcal{U}] = M[\mathcal{U}']$.

$$\mathcal{U}' = \{W \in \mathcal{P}(\mathbb{N}) \cap M : \psi^{-1}(W) \in \mathcal{U}\},$$
$$\mathcal{U} = \{Z \in [\mathbb{N}]^\infty \cap M : \psi``(Z \cap Y) \in \mathcal{U}'\},$$

where $Y \in \mathcal{U}$ and $\psi \restriction Y$ is injective. □

$$\mathcal{V} := f``(G)^{\uparrow}, \quad \mathcal{U} := \pi``(G)^{\uparrow}, \quad \mathcal{U}' := h``(G)^{\uparrow} = (\max^*)``(\mathcal{V})^{\uparrow}.$$

Therefore, we have that $\mathcal{V} \in M[\mathcal{U}'] = M[(\max^*)``(\mathcal{V})^{\uparrow}]$. Let $\dot{\mathcal{V}}$ be the canonical name for a $(\mathbf{FIN}^{[\infty]}, \leq)$-generic filter. Then there exists some $P \in \mathbf{FIN}^{[\infty]}$ such that for all $(\mathbf{FIN}^{[\infty]}, \leq)$-generic filter $\mathcal{V}'$ containing P:

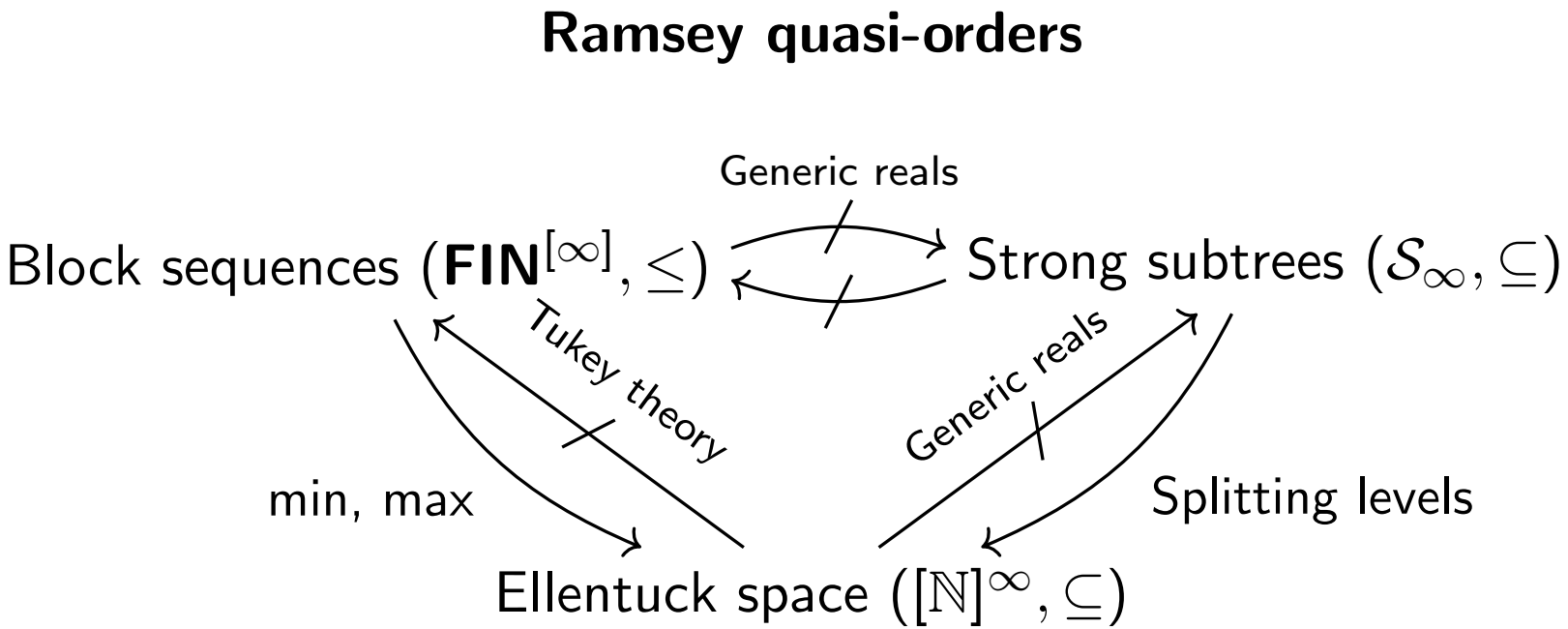
$$\mathcal{V}' = i_{(\max^*)``(\mathcal{V}')}(\dot{\mathcal{V}}).$$

Let $P_0, P_1 \leq P$ be incompatible such that $\max^*(P_0) = \max^*(P_1)$. There exist $\mathcal{V}_0, \mathcal{V}_1$ $(\mathbf{FIN}^{[\infty]}, \leq)$-generic such that $P_0 \in \mathcal{V}_0$ and $P_1 \in \mathcal{V}_1$ and $(\max^*)``(\mathcal{V}_0) = (\max^*)``(\mathcal{V}_1)$. Then:

$$\mathcal{V}_0 = i_{(\max^*)``(\mathcal{V}_0)}(\dot{\mathcal{V}}) = i_{(\max^*)``(\mathcal{V}_1)}(\dot{\mathcal{V}}) = \mathcal{V}_1,$$

a contradiction. □

Yay!


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Open problems

Let $\mathbf{FIN}_k^{[\infty]}$ be the Gowers space of infinite block sequences.

Theorem (Unger-Y., 2026)

There is no weak projection $(\mathbf{FIN}^{[\infty]}, \leq) \to (\mathbf{FIN}_2^{[\infty]}, \leq)$.

Problem

Is there a weak projection $(\mathbf{FIN}_k^{[\infty]}, \leq) \to (\mathbf{FIN}_{k+1}^{[\infty]}, \leq)$ for $k \geq 2$?

Let $\mathcal{E}_\infty$ be the Carlson–Simpson space of equivalence relations on $\mathbb{N}$.

Theorem (Unger-Y.,2026)

Assuming large cardinals, there is no weak projection $(\mathbf{FIN}_k^{[\infty]}, \leq) \to (\mathcal{E}_\infty, \leq)$ *for* $k \geq 2$.

More precisely, we used $\mathbf{\Sigma}^1_4$ generic absoluteness, as "there is a weak projection" is a $\mathbf{\Pi}^1_4$ statement.

Problem

Is the large cardinal assumption in the theorem above necessary?

Thanks for listening!

A **weak projection** is a monotone Alaoglu–Birkhoff convergent map. For monotone maps:

- Alaoglu–Birkhoff convergent $\iff$ transfers generic filters.
- For the three Ramsey quasi-orders below, Alaoglu–Birkhoff convergent $\iff$ quasi-open ("$\iff$" transfers analytic Ramsey property).

