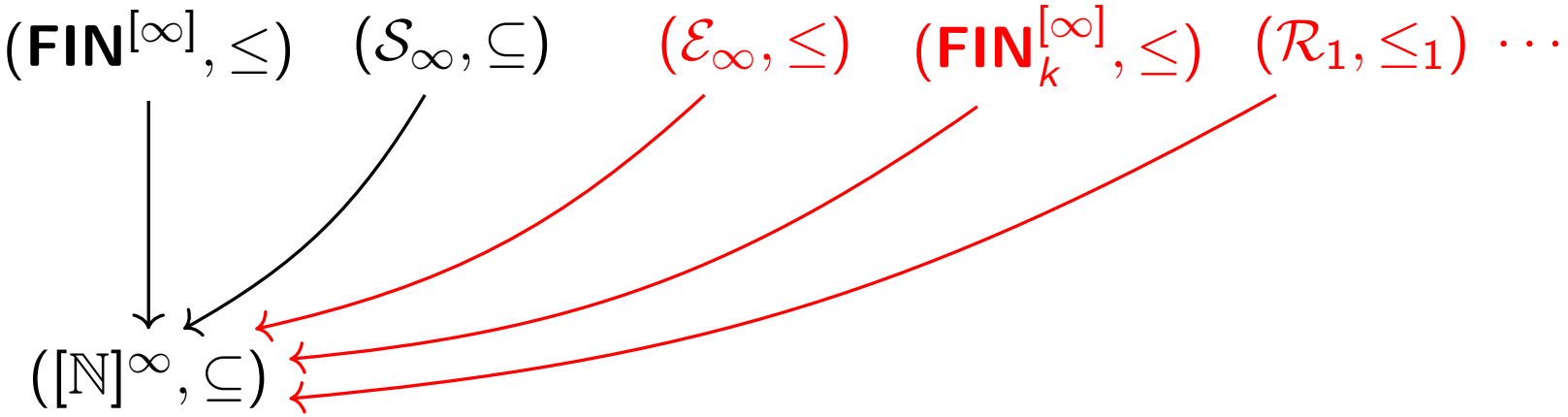
Projections between Ramsey quasi-orders and topological Ramsey spaces (Part 2)

Clement Yung
Joint work with Spencer Unger

13 Nov 2026

Recap

Ramsey quasi-orders



Question. Do all topological Ramsey spaces project into $([\mathbb{N}]^\infty, \subseteq)$?

$(\mathbf{FIN}^{[\infty]}, \leq)$	Block sequences
$(\mathcal{S}_\infty, \subseteq)$	Strong subtrees
$(\mathcal{E}_\infty, \leq)$	Carlson–Simpson
$(\mathbf{FIN}_k^{[\infty]}, \leq)$	Gowers block seqs

Topological Ramsey spaces

Last week, I mentioned that $([\mathbb{N}]^\infty, \subseteq)$, $(\mathbf{FIN}^{[\infty]}, \leq)$ and $(\mathcal{S}_\infty, \subseteq)$ are topological Ramsey spaces. **But what are topological Ramsey spaces anyway?**

Topological Ramsey spaces are Ramsey quasi-orders with the following additional structures:

- **(Sequencing)** Every $A \in \mathcal{R}$ is an infinite sequence.
- **(Finitisation)** The quasi-order $\leq$ is induced by some quasi-order on the set of finite sequences.
- **(Amalgamation)** We can always build an infinite substructure of some $A \in \mathcal{R}$ from some finite substructure.

(+ a lot more that I won't talk about)

Sequencing

Topological Ramsey spaces are Ramsey quasi-orders with a sequence structure:

Ramsey quasi-order
$(\mathcal{R}, \leq)$

Topological Ramsey space
$(\mathcal{R}, \leq, r)$
$r : \mathcal{R} \times \omega \to \mathcal{AR}$

$\mathcal{AR}$ is the set of finite sequences, and each infinite sequence $A \in \mathcal{R}$ is mapped to $r_n(A) := r(A, n)$, the first n elements of A.

Finitisation

Additionally, $\leq$ is induced by some quasi-order on $\leq_{\mathrm{fin}}$ on $\mathcal{AR}$, and satisfies the following:

1. ("Locally finite") Every finite structure has only finitely many substructures.

$$\{b : b \leq_{\mathrm{fin}} a\} \text{ is finite for all } a \in \mathcal{AR}.$$

2. Substructure iff every finite substructure is a substructure.

$$B \leq A \iff \forall n \exists m\, r_n(B) \leq_{\mathrm{fin}} r_m(A).$$

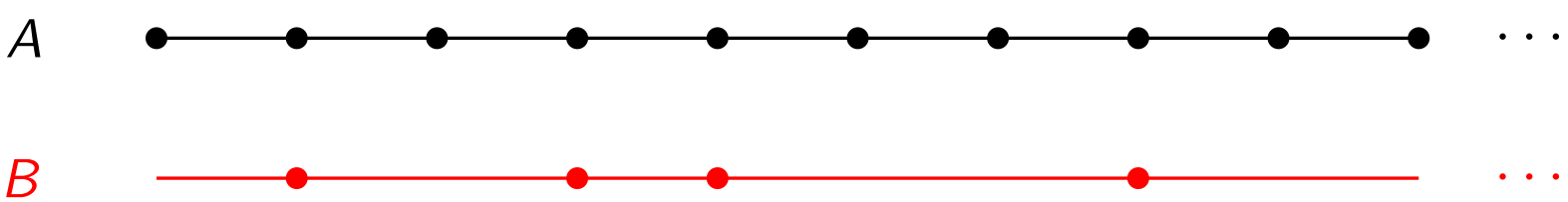
3. Initial segment of substructure $=$ substructure of initial segment.

If $d \leq_{\mathrm{fin}} b$ and $c \sqsubseteq b$, then $c \leq_{\mathrm{fin}} a$ for some $a \sqsubseteq b$.

Amalgamation

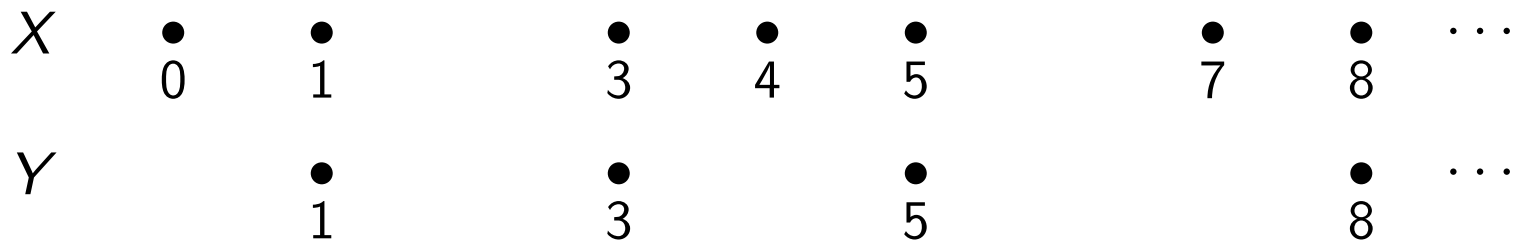
One part of the amalgamation assumption allows us to build an infinite substructure of some $A \in \mathcal{R}$ by inductively extending finite substructures.

Topological Ramsey space $(\mathcal{R}, \leq, r)$



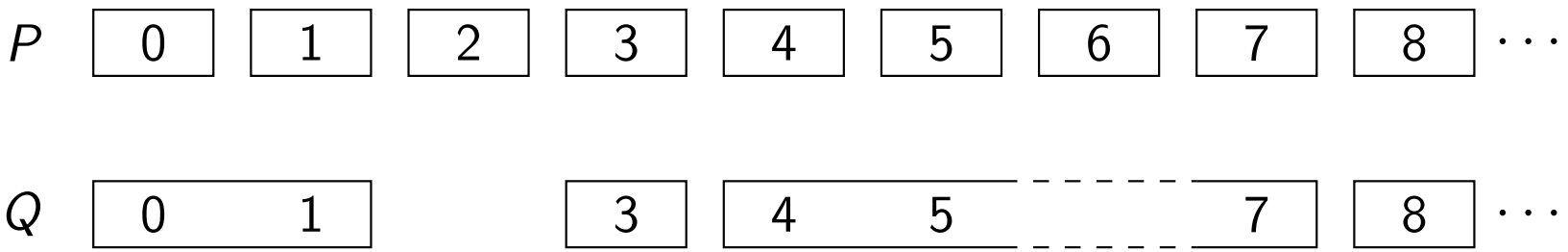
$B \leq A$ (because $\forall n \, \exists m \, r_n(B) \leq_{\text{fin}} r_m(A)$)

Ellentuck space $([\mathbb{N}]^\infty, \subseteq, r)$



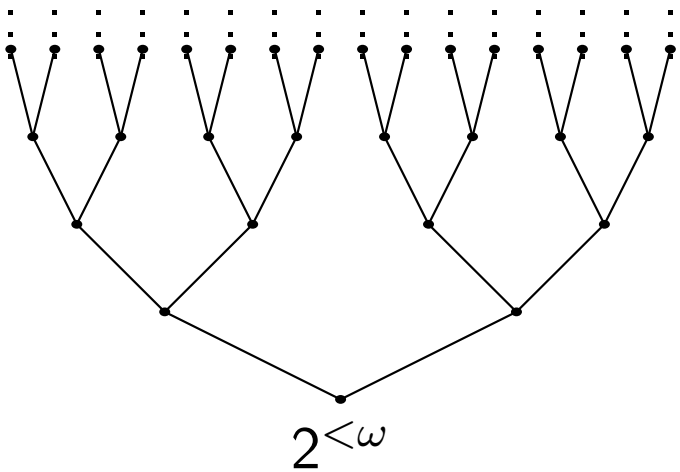
- ✓ **Sequencing** ($r_n(X) =$ first n numbers)
- ✓ **Finitisation** (witnessed by $\leq_{\text{fin}}$ being $\subseteq$)
- ✓ **Amalgamation**

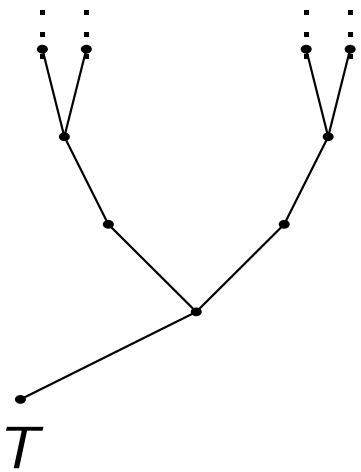
Block sequences $(\mathbf{FIN}^{[\infty]}, \leq, r)$



- ✓ **Sequencing** ($r_n(P) =$ first n blocks)
- ✓ **Finitisation** (witnessed by $a \leq_{\mathrm{fin}} b$ iff $\langle a \rangle \subseteq \langle b \rangle$)
- ✓ **Amalgamation**

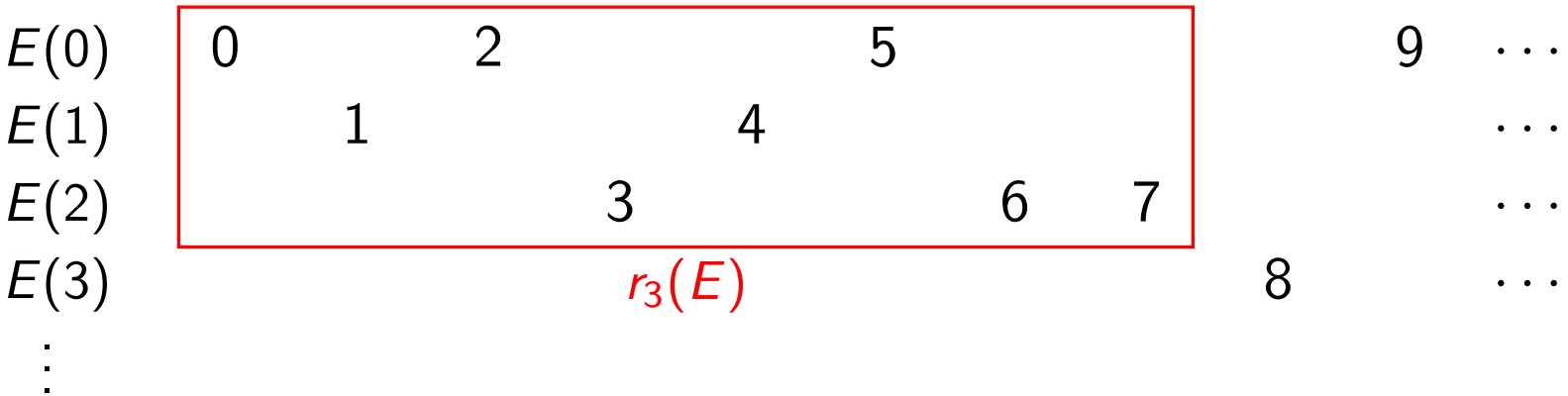
Strong subtrees $(\mathcal{S}_\infty, \subseteq, r)$





- ✓ **Sequencing** ($r_n(T) =$ first n splitting levels)
- ✓ **Finitisation** (witnessed by $a \leq_{\mathrm{fin}} b$ iff $a \subseteq b$ and each terminal node of a splits or is terminal in b)
- ✓ **Amalgamation**

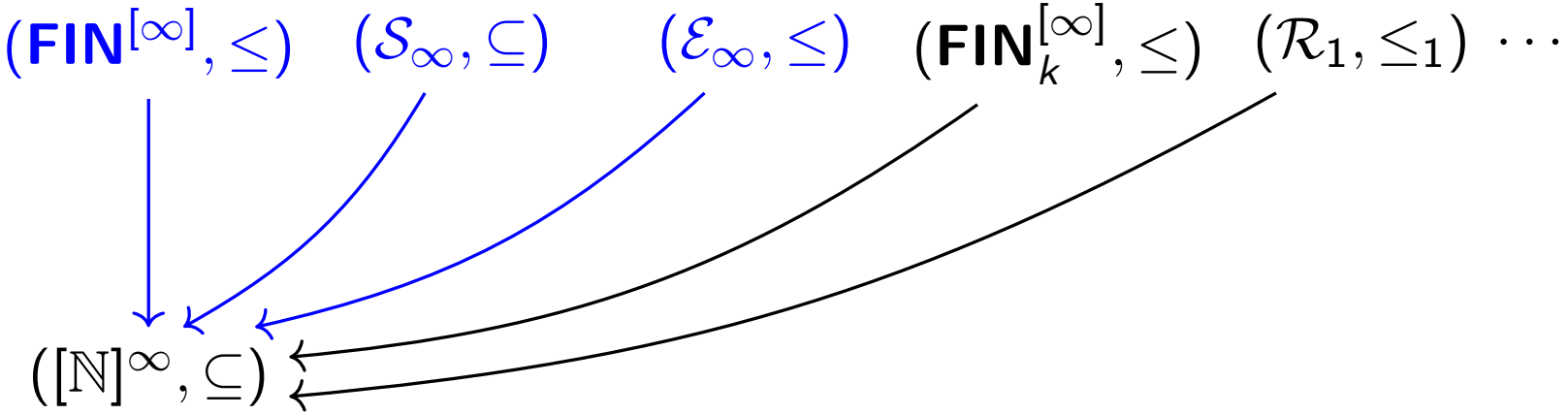
Carlson–Simpson space $(\mathcal{E}_\infty, \leq)$



- ✓ **Sequencing** ($r_n(E) = E \restriction p_n$, where p_n is the n^{th}-least representative)
- ✓ **Finitisation** (witnessed by $a \leq_{\text{fin}} b$ iff a coarsens b and $\text{dom}(a) = \text{dom}(b)$)
- ✓ **Amalgamation**

Projection to Ellentuck space

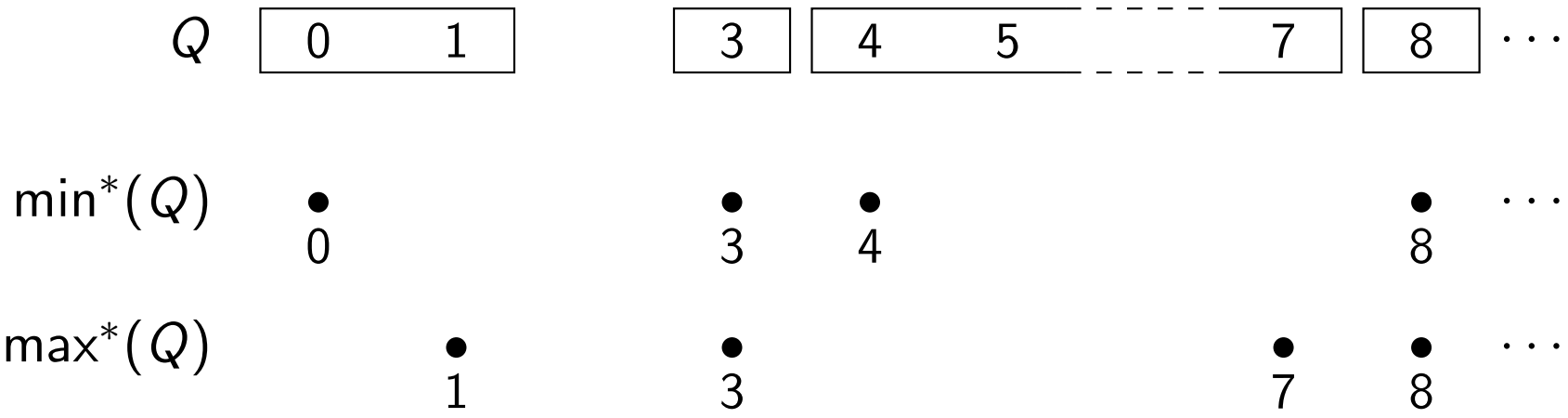
Ramsey quasi-orders



Question. How do these spaces project into $([\mathbb{N}]^\infty, \subseteq)$?

$(\mathbf{FIN}^{[\infty]}, \leq)$	Block sequences
$(\mathcal{S}_\infty, \subseteq)$	Strong subtrees
$(\mathcal{E}_\infty, \leq)$	Carlson–Simpson
$(\mathbf{FIN}_k^{[\infty]}, \leq)$	Gowers block seqs

Projection $(\mathbf{FIN}^{[\infty]}, \leq, r) \to ([\mathbb{N}]^\infty, \subseteq, r)$



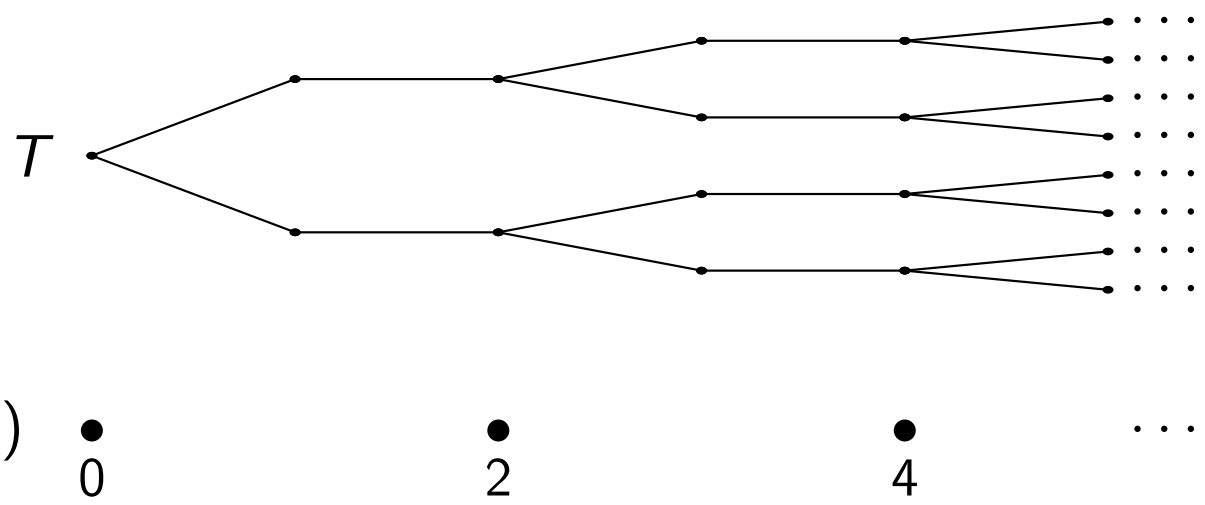
Lemma

The maps $\min^*, \max^* : \mathbf{FIN}^{[\infty]} \to [\mathbb{N}]^\infty$ *defined by:*

$$\min^*((p_n)_{n<\omega}) = \{\min(p_n) : n < \omega\},$$
$$\max^*((p_n)_{n<\omega}) = \{\max(p_n) : n < \omega\}.$$

are cofinal continuous monotone, and $f``[P] = [f(P)]$ *for all* P.

Projection $(\mathcal{S}_\infty, \subseteq, r) \to ([\mathbb{N}]^\infty, \subseteq, r)$



Lemma

The map $\pi : \mathcal{S}_\infty \to [\mathbb{N}]^\infty$ defined by:

$$\pi(T) := \{\mathrm{dom}(s) : s \in T \text{ is a splitting node}\}$$

is cofinal continuous monotone, and π"$[T] = [\pi(T)]$ for all T.

Projection $(\mathcal{E}_\infty, \leq, r) \to ([\mathbb{N}^+]^\infty, \subseteq, r)$

$E(0)$	~~0~~		2			5		9	$\cdots$
$E(1)$		1			4				$\cdots$
$E(2)$				3			6 7		$\cdots$
$E(3)$								8	$\cdots$
$\vdots$									
$p^*(E)$		1		3				8	$\cdots$

Lemma

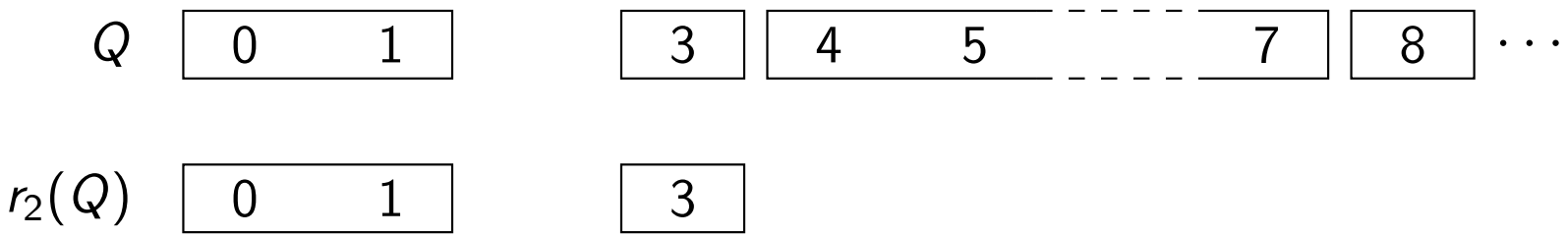
The map $p^* : \mathcal{E}_\infty \to [\mathbb{N}^+]^\infty$ *defined by:*

$$p^*(E) := \{\min(E(n)) : n \in \mathbb{N}^+\}$$

is cofinal continuous monotone, and $p^*[E] = [p^*(E)]$ *for all* E.

Observation 1 of 2. We can define these maps on finite seqs.

Projection $(\mathbf{FIN}^{[\infty]}, \leq, r) \to ([\mathbb{N}]^\infty, \subseteq, r)$



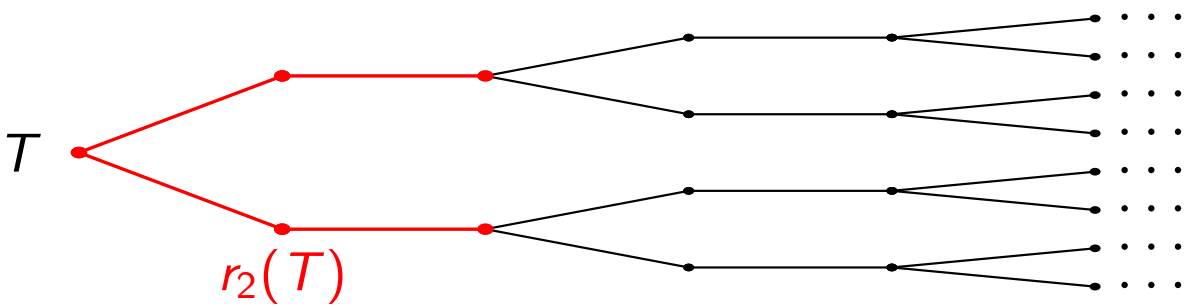
$$\overline{\max}^*(a) = \{\max(x_n) : n < N\}, \text{ where } a = (x_n)_{n<N}$$

$$\max^*(Q) = \{1, 3, 7, 8, \dots\}, \quad \overline{\max}^*(r_2(Q)) = \{1, 3\}$$

$$\max^*(Q) = \bigcup_{n<\omega} \overline{\max}^*(r_n(Q))$$

Observation 1 of 2. We can define these maps on finite seqs.

Projection $(\mathcal{S}_\infty, \subseteq, r) \to ([\mathbb{N}]^\infty, \subseteq, r)$



$$\overline{\pi}(a) := \{\mathrm{dom}(s) : s \in a \text{ splits or is terminal}\}$$

$$\pi(T) = \{0, 2, 4, \dots\}, \quad \overline{\pi}(r_2(T)) = \{0, 2\}$$

$$\pi(T) = \bigcup_{n<\omega} \overline{\pi}(r_n(T))$$

Observation 1 of 2. We can define these maps on finite seqs.

Projection $(\mathcal{E}_\infty, \leq, r) \to ([\mathbb{N}^+]^\infty, \subseteq, r)$

$E(0)$	0		2			5			9	$\cdots$
$E(1)$		1			4					$\cdots$
$E(2)$				3			6	7		$\cdots$
$E(3)$									8	$\cdots$
$\vdots$										

$r_3(E)$

$$\overline{p}^*(a) = \{\min(a(n)) : 0 < n < \#\text{classes}\} \cup \{\text{dom}(a)\}$$

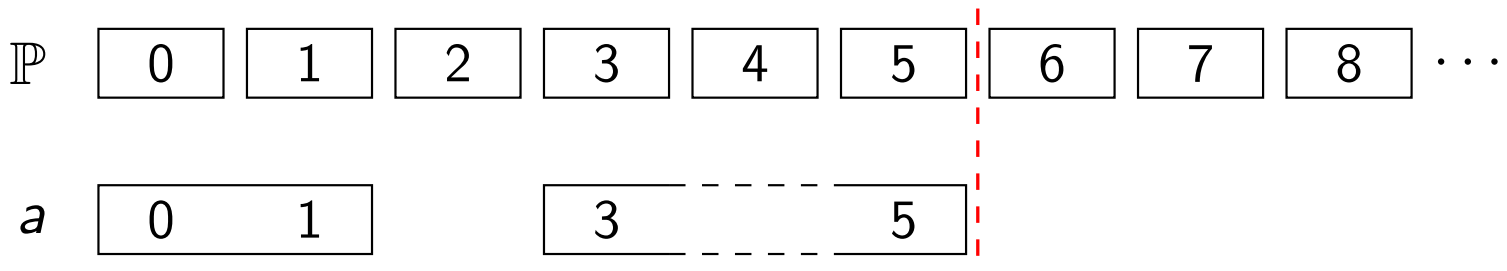
$$p^*(E) = \{1, 3, 8, \dots\}, \quad \overline{p}^*(r_3(E)) = \{1, 3, 8\}$$

$$p^*(E) = \bigcup_{0<n<\omega} \overline{p}^*(r_n(E))$$

Observation 2 of 2. These projections are defined in the same way. If $\mathbb{A} \in \mathcal{R}$ is $\leq$-maximal and $a \in \mathcal{AR}$, then let:

$$\mathrm{depth}(a) := \min\{m : a \leq_{\mathrm{fin}} r_m(\mathbb{A})\}.$$

Projection $(\mathbf{FIN}^{[\infty]}, \leq, r) \to ([\mathbb{N}]^\infty, \subseteq, r)$



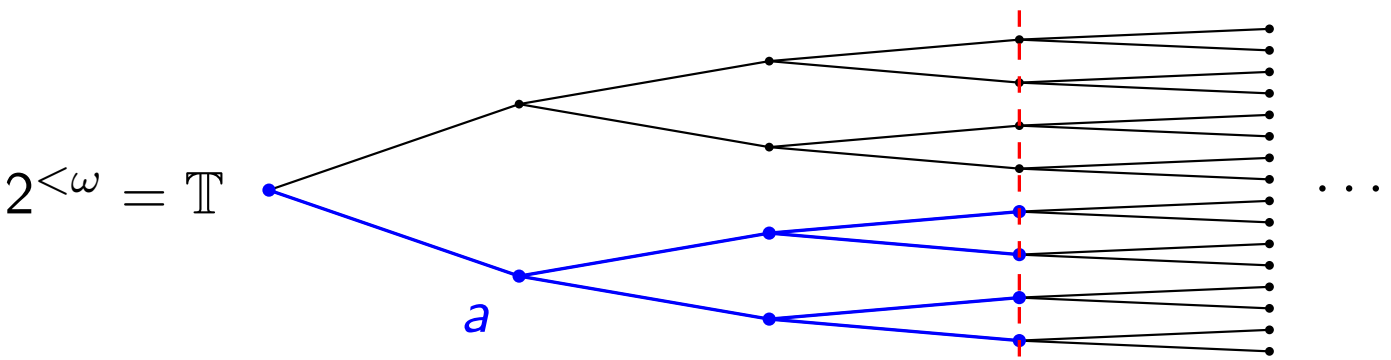
$$\mathrm{depth}(a) = 5 + 1$$

$$\overline{\max}^*(a) = \{1, 5\}$$

Observation 2 of 2. These projections are defined in the same way. If $\mathbb{A} \in \mathcal{R}$ is $\leq$-maximal and $a \in \mathcal{AR}$, then let:

$$\mathrm{depth}(a) := \min\{m : a \leq_{\mathrm{fin}} r_m(\mathbb{A})\}.$$

Projection $(\mathcal{S}_\infty, \subseteq, r) \to ([\mathbb{N}]^\infty, \subseteq, r)$



$$\mathrm{depth}(a) = 3 + 1$$

$$\overline{\pi}(a) = \{1, 2, 3\}$$

Observation 2 of 2. These projections are defined in the same way. If $\mathbb{A} \in \mathcal{R}$ is $\leq$-maximal and $a \in \mathcal{AR}$, then let:

$$\mathrm{depth}(a) := \min\{m : a \leq_{\mathrm{fin}} r_m(\mathbb{A})\}.$$

Projection $(\mathcal{E}_\infty, \leq, r) \to ([\mathbb{N}^+]^\infty, \subseteq, r)$

$\mathbb{E}(0)$ 0

$\mathbb{E}(1)$ 1

$\mathbb{E}(2)$ 2

$\mathbb{E}(3)$ 3

$\vdots$ $\ddots$

$a(0)$ 0 3

$a(1)$ 1 2

$\mathrm{depth}(a) = 4$

$\overline{p}^*(a) = \{1, 4\}$

(no $+1$ as codomain is $[\mathbb{N}^+]^\infty$)

Observation 2 of 2. These projections are defined in the same way. If $\mathbb{A} \in \mathcal{R}$ is $\leq$-maximal and $a \in \mathcal{AR}$, then let:

$$\mathrm{depth}(a) := \min\{m : a \leq_{\mathrm{fin}} r_m(\mathbb{A})\}.$$

If $(\mathcal{R}, \leq, r)$ is one of these topological Ramsey spaces, and $f : \mathcal{R} \to [\mathbb{N}]^\infty$ is any of these projections, then:

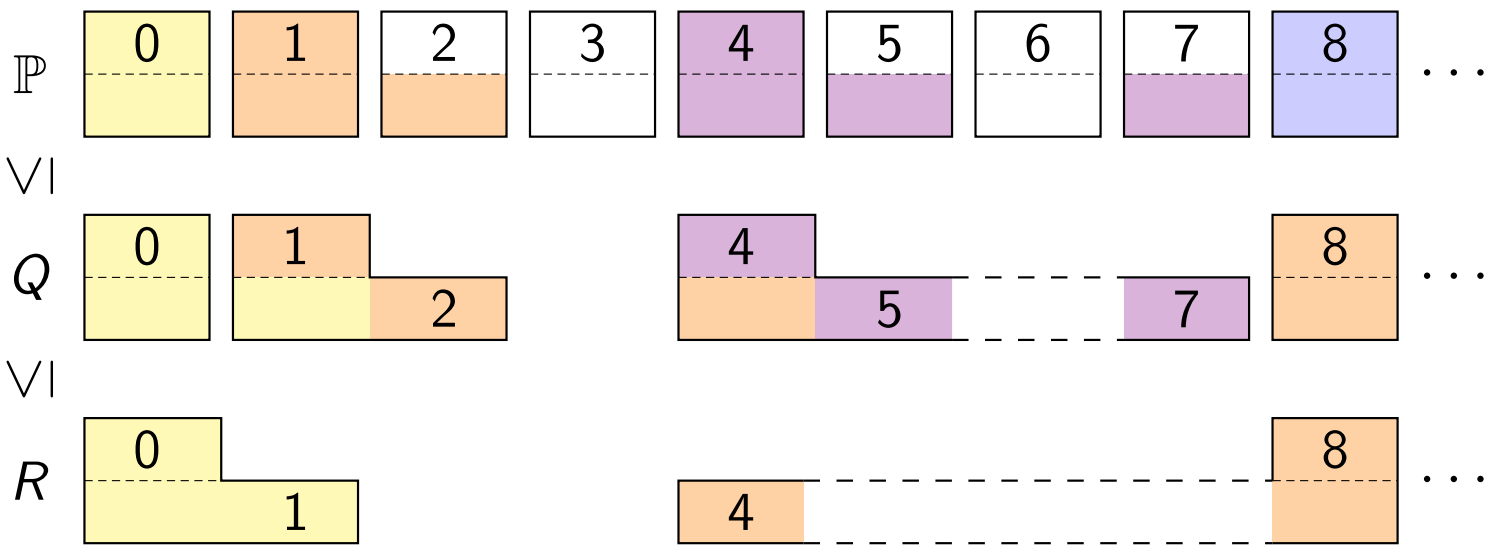
$$f(A) = \{\mathrm{depth}(r_n(A)) - 1 : n > 0\},$$

Question. For a general topological Ramsey spaces with some $\leq$-maximal element $\mathbb{A}$, is the above map cofinal continuous monotone quasi-open?

$$f(A) := \{\mathrm{depth}(r_n(A)) - 1 : n > 0\}.$$

f is generally **not** monotone.

Gowers block seqs $(\mathbf{FIN}_2^{[\infty]}, \leq, r)$



$$f(Q) = \{0, 2, 7, 8, \dots\}, \quad f(R) = \{1, 8, \dots\}$$

In general, if $A \in \mathcal{R}$ and $a \in \mathcal{AR}{\restriction}A$ (i.e. a is a finite substructure of A), we can define:

$$\mathrm{depth}_A(a) := \min\{m : a \leq_{\mathrm{fin}} r_m(A)\}.$$

If $B \leq A$ and $a \in \mathcal{AR}{\restriction}B$, then we have:

$$\mathrm{depth}_A(a) \leq \mathrm{depth}_A(r_{\mathrm{depth}_B(a)}(B)).$$

Equality doesn't always hold (e.g. $\mathbf{FIN}_2$ example), but it always holds for $[\mathbb{N}]^\infty$, $\mathbf{FIN}^{[\infty]}$, $\mathcal{S}_\infty$ and $\mathcal{E}_\infty$.

Definition

$(\mathcal{R}, \leq, r)$ is *depth-preserving* if for all $A, B \in \mathcal{R}$ and $a \in \mathcal{AR}{\restriction}B$ such that $B \leq A$:

$$\mathrm{depth}_A(a) = \mathrm{depth}_A(r_{\mathrm{depth}_B(a)}(A)).$$

Lemma

If $(\mathcal{R}, \leq, r)$ is depth-preserving, then the map $f : \mathcal{R} \to [\mathbb{N}]^\infty$ defined by:

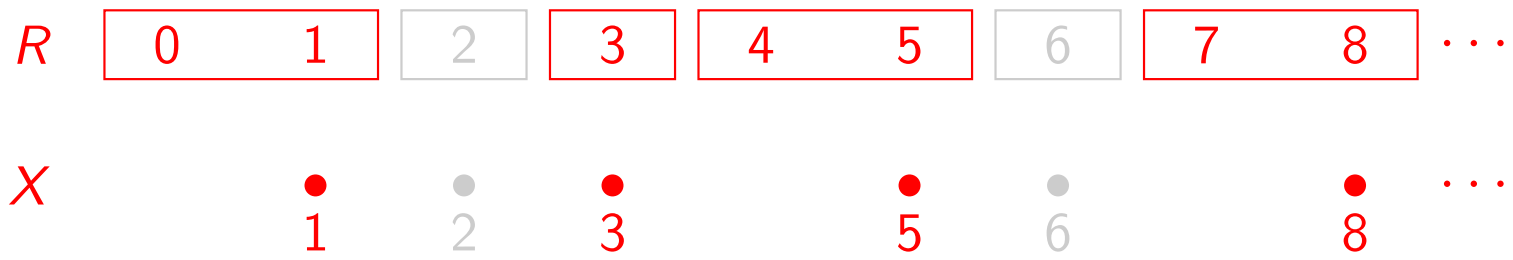
$$f(A) := \{\text{depth}(r_n(A)) - 1 : 0 < n < \omega\}$$

is (continuous) monotone.

What about cofinal quasi-open?

Let's recall why f is cofinal quasi-open for $\mathbf{FIN}^{[\infty]} \to [\mathbb{N}]^\infty$.

Projection $(\mathbf{FIN}^{[\infty]}, \leq, r) \to ([\mathbb{N}]^\infty, \subseteq, r)$



Lemma

The map $\max^* : \mathbf{FIN}^{[\infty]} \to [\mathbb{N}]^\infty$ *defined by:*

$$\max^*((p_n)_{n<\omega}) = \{\max(p_n) : n < \omega\}$$

is cofinal continuous monotone, and $(\max^*)``[P] = [\max^*(P)]$ *for all* P.

The "pick the next right element one-by-one" also works for $\mathcal{S}_\infty$ and $\mathcal{E}_\infty$. **What is actually happening here?**

Notation. If $A \in \mathcal{R}$ and $a \in \mathcal{AR}{\restriction}A$, then let:

$$r_{\mathrm{lh}(a)+1}[a, A] := \{b \in \mathcal{AR}{\restriction}A : a \sqsubset b \text{ and } \mathrm{lh}(b) = \mathrm{lh}(a) + 1\}.$$

We write:

$$D_{a,A} = \{\mathrm{depth}_A(b) - \mathrm{depth}_A(a) - 1 : b \in r_{\mathrm{lh}(a)+1}[a, A]\}.$$

I'll explain why there's a -1 there later.

Observation. For $[\mathbb{N}]^\infty$, $\mathbf{FIN}^{[\infty]}$, $\mathcal{S}_\infty$ and $\mathcal{E}_\infty$, $D_{a,A} = \mathbb{N}$ for all a, A, i.e. you can always pick the next element of arbitrary depth.

Extend f to finite sequences by:

$$\overline{f}(a) := \{\mathrm{depth}(r_n(a)) - 1 : 0 < n \leq \mathrm{lh}(a)\}.$$

Theorem

If $(\mathcal{R}, \leq, r)$ is depth-preserving and $D_{a,A} = \mathbb{N}$ for all $A \in \mathcal{R}$ and $a \in \mathcal{AR}{\restriction}A$, then the map:

$$f(A) := \{\mathrm{depth}(r_n(A)) - 1 : 0 < n < \omega\}$$

satisfies the following:

1. 1.1 *If $a \sqsubset b$, then $\overline{f}(a) \sqsubset \overline{f}(b)$.*
   1.2 *If $a \leq_{\mathrm{fin}} b$, then $\overline{f}(a) \subseteq \overline{f}(b)$.*
2. *For all $a \in \mathcal{AR}$, $\mathrm{lh}(\overline{f}(a)) = \mathrm{lh}(a)$.*
3. *$f(\mathbb{A}) = \mathbb{N}$.*
4. *For all $A \in \mathcal{R}$ and $a \in \mathcal{AR}{\restriction}A$, $f\text{“}[a, A] = [\overline{f}(a), f(A)]$.*

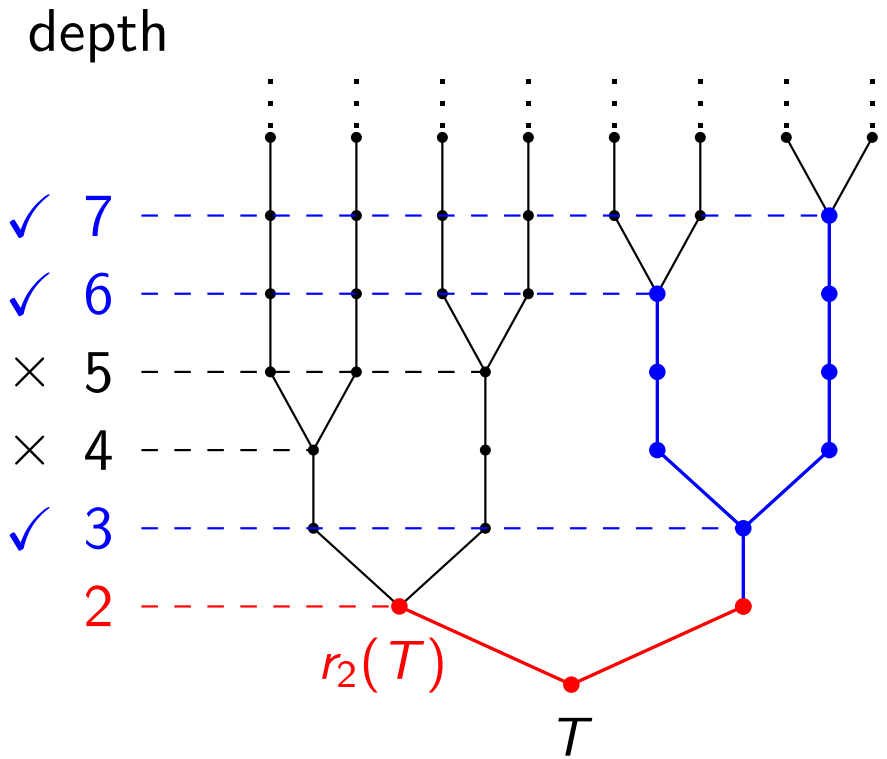
Depth-uniform property

Question. Is it necessary to assume the depth-preserving property, and $D_{a,A} = \mathbb{N}$ for all a, A?

Partial answer. At the very least, we can't remove them completely.

Sacks forcing $(\mathbb{S}, \subseteq, r)$ is a topological Ramsey space that does not weakly project into $([\mathbb{N}]^\infty, \subseteq, r)$ (due to forcing minimality).

Sacks forcing $(\mathbb{S}, \subseteq, r)$



$$D_{r_2(T),T} = \{0, 3, 4, \dots\}$$

But we can slightly weaken the hypothesis $D_{a,A} = \mathbb{N}$ for all a, A.

Definition

$(\mathcal{R}, \leq, r)$ is *depth-uniform* if for all $A, B \in \mathcal{R}$, $a \in \mathcal{AR}{\restriction}A$ and $b \in \mathcal{AR}{\restriction}B$, we have $D_{a,A} = D_{b,B}$. We call this set $D(\mathcal{R})$.

Dichotomy theorem

If $(\mathcal{R}, \leq, r)$ is depth-preserving depth-uniform with a $\leq$-maximal element $\mathbb{A}$, then precisely one of the following hold:

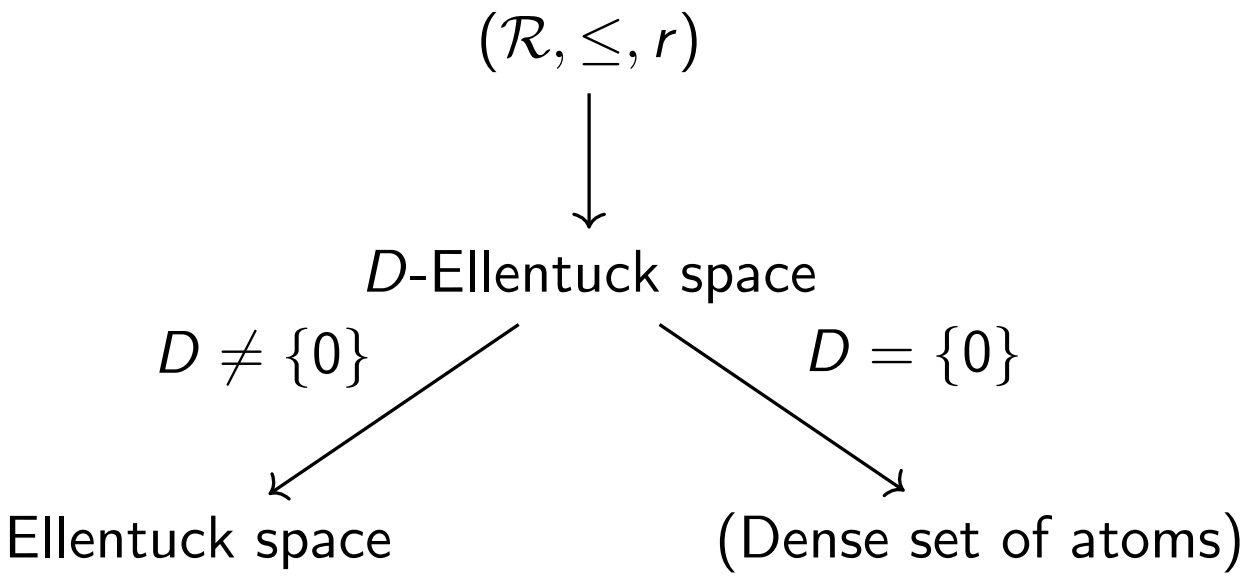
1. *There is some $f : \mathcal{R} \to [\mathbb{N}]^\infty$ such that for all $A \in \mathcal{R}$ and $a \in \mathcal{AR}{\restriction}A$, the restricted map:*

$$f{\restriction}[a, A] : [a, A] \to [\overline{f}(a), f(A)]$$

   is (cofinal continuous) monotone quasi-open.

2. *$(\mathcal{R}, \leq)$ has a dense set of atoms.*

Starting with a depth-preserving depth-uniform topological Ramsey space $(\mathcal{R}, \leq, r)$ with a $\leq$-maximal element and $D := D(\mathcal{R})$, we prove the dichotomy theorem this way:



D-Ellentuck space

Topological Ramsey spaces such that $D_{a,A} = \mathbb{N}$ for all a, A are easy to come by. Meanwhile, Sacks forcing $(\mathbb{S}, \subseteq, r)$ fails this, but it is not depth-uniform.

Question. Given some $D \subseteq \mathbb{N}$, is there some depth-uniform topological Ramsey space with $D(\mathcal{R}) = D$?

Fix some $D \subseteq \mathbb{N}$. Given $A = \{m_n : n < |A|\} \subseteq \mathbb{N}$ and $B \subseteq A$, we may write $B = \{m_{n_k} : k < |B|\}$, where (n_k) is increasing.

Notation. We write $B \subseteq^D A$ iff all of the following hold:

- $B \subseteq A$.
- $n_0 \in D$.
- $n_{k+1} - n_k - 1 \in D$ for all k.

Note that if $D = \mathbb{N}$, then $\subseteq^D$ is just $\subseteq$.

Lemma

The following are equivalent:

1. $([\mathbb{N}]^{\leq\infty}, \subseteq^D)$ *is a partial order.*
2. D *is a* submonoid (of $(\mathbb{N}, +)$), *i.e.* $0 \in D$ *and for all* $a, b \in D$, $a + b \in D$.

Fix some submonoid $D \subseteq \mathbb{N}$. For each $A \in [\mathbb{N}]^\infty$, write:

$$[A]^{\infty,D} := \{B \in [A]^\infty : B \subseteq^D A\},$$

and define $[A]^{<\infty,D}$ similarly.

Theorem

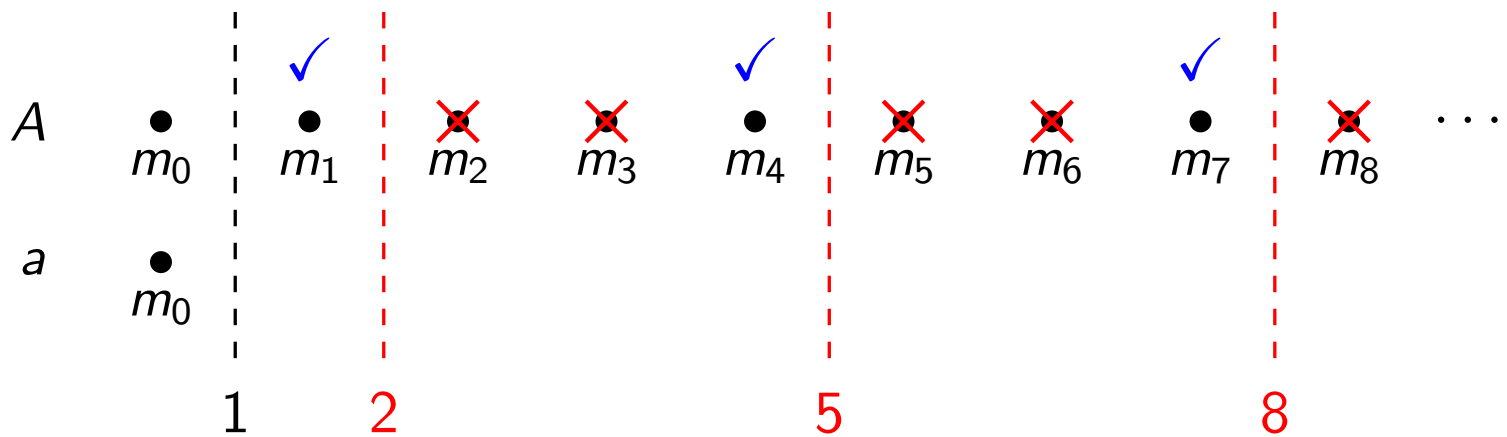
$([\mathbb{N}]^{\infty,D}, \subseteq^D, r)$ is a depth-preserving depth-uniform topological Ramsey space such that $D([\mathbb{N}]^{\infty,D}) = D$.

We discuss why it is depth-uniform and $D([\mathbb{N}]^{\infty,D}) = D$, and satisfies the pigeonhole property (which implies the analytic Ramsey property) and why $D([\mathbb{N}]^{\infty,D}) = D$.

Depth-uniform property. Recall that for $A \in [\mathbb{N}]^{\infty,D}$ and $a \in [A]^{<\infty,D}$:

$$D_{a,A} = \{\mathrm{depth}_A(b) - \mathrm{depth}_A(a) - 1 : b \in r_{\mathrm{lh}(a)+1}[a, A]\}.$$

D-Ellentuck space $([\mathbb{N}]^{\infty,D}, \subseteq^D, r)$, $D = \{0, 3, 6, \dots\}$



$$D_{a,A} = \{2 - 1 - 1, 5 - 1 - 1, 8 - 1 - 1, \dots\} = \{0, 3, 6, \dots\}$$

Pigeonhole property. The pigeonhole property of $[\mathbb{N}]^{\infty,D}$ reduces to the following statement:

$(*)$ For all $A \in [\mathbb{N}]^{\infty,D}$ and $\mathcal{O} \subseteq D$, there exists some $B \in [A]^{\infty,D}$ such that $B \cap D \subseteq \mathcal{O}$ or $B \cap D \subseteq \mathcal{O}^c$.

Key fact. If $X \subseteq \mathbb{N}$, write $\gcd(X)$ to be the greatest integer dividing every element of X.

Fact

If D is a submonoid and $g := \gcd(D)$, then D is a cofinite subset of $g\mathbb{N}$ (i.e. $D \subseteq g\mathbb{N}$ and $g\mathbb{N} \setminus D$ is finite).

($*$) For all $A \in [\mathbb{N}]^{\infty,D}$ and $\mathcal{O} \subseteq D$, there exists some $B \in [A]^{\infty,D}$ such that $B \cap D \subseteq \mathcal{O}$ or $B \cap D \subseteq \mathcal{O}^c$.

Fact. D is cofinite subset of $g\mathbb{N}$, where $g := \gcd(D)$.

Case 1 — $D = \{0\}$: $[\mathbb{N}]^{\infty,D} = \{\mathbb{N}\}$, so we're done.

Case 2 — $A \cap \mathcal{O}$ finite: Let $A = \{m_n : n < \omega\}$, and let $N > g\mathbb{N} \setminus D$. Observe that, for any $k \geq N$:

$$B_k := \{m_n : n \geq kg\} \in [A]^{\infty,D}.$$

Let $k \geq N$ be large enough so that $m_{kg} > \max(A \cap \mathcal{O})$. Then $B_k \cap \mathcal{O} = \emptyset$.

($*$) For all $A \in [\mathbb{N}]^{\infty,D}$ and $\mathcal{O} \subseteq D$, there exists some $B \in [A]^{\infty,D}$ such that $B \cap D \subseteq \mathcal{O}$ or $B \cap D \subseteq \mathcal{O}^c$.

Fact. D is cofinite subset of $g\mathbb{N}$, where $g := \gcd(D)$.

Case 3 — $A \cap \mathcal{O}$ infinite: Let $A = \{m_n : n < \omega\}$, and let $N > g\mathbb{N} \setminus D$. We may assume that $m_0 \geq N$. Observe that for all n:

$$m_n \equiv n \pmod{g}$$

Consequently:

1. $m_n \in D$ iff $n = kg$ for some $k \in \mathbb{N}$.
2. $n_i - n_j - 1 \in D$ iff $n_i - n_j \equiv 1 \pmod{g}$.

Therefore, there is some $(k_i)_{i<\omega}$ increasing such that:

$$A \cap \mathcal{O} = \{m_{k_0 g}, m_{k_1 g}, m_{k_2 g}, \dots\}.$$

$\vdots$

Consequently:

1. $m_n \in D$ iff $n = kg$ for some $k \in \mathbb{N}$.
2. $m_{n_i} - m_{n_j} - 1 \in D$ iff $n_i - n_j \equiv 1 \pmod{g}$.

Therefore, there is some $(k_i)_{i<\omega}$ increasing such that:

$$A \cap \mathcal{O} = \{m_{k_0 g}, m_{k_1 g}, m_{k_2 g}, \dots\}.$$

But $A \cap \mathcal{O} \notin [\mathbb{N}]^{\infty, D}$! So we "pad" the gaps:

$$B := \{m_{k_0 g}, m_{k_0 g+1}, \dots, m_{k_0 g+g-1}, m_{k_1 g}, m_{k_1 g+1}, \dots\}.$$

Then $B \subseteq^D A$ and $B \cap D = A \cap \mathcal{O} \subseteq \mathcal{O}$. □

When $D \neq \{0\}$

Theorem

If $D \neq \{0\}$, then there is some $f : [\mathbb{N}]^{\infty,D} \to [\mathbb{N}]^{\infty}$ such that for all $A \in [\mathbb{N}]^{\infty,D}$ and $a \in [A]^{<\infty,D}$, the restricted map:

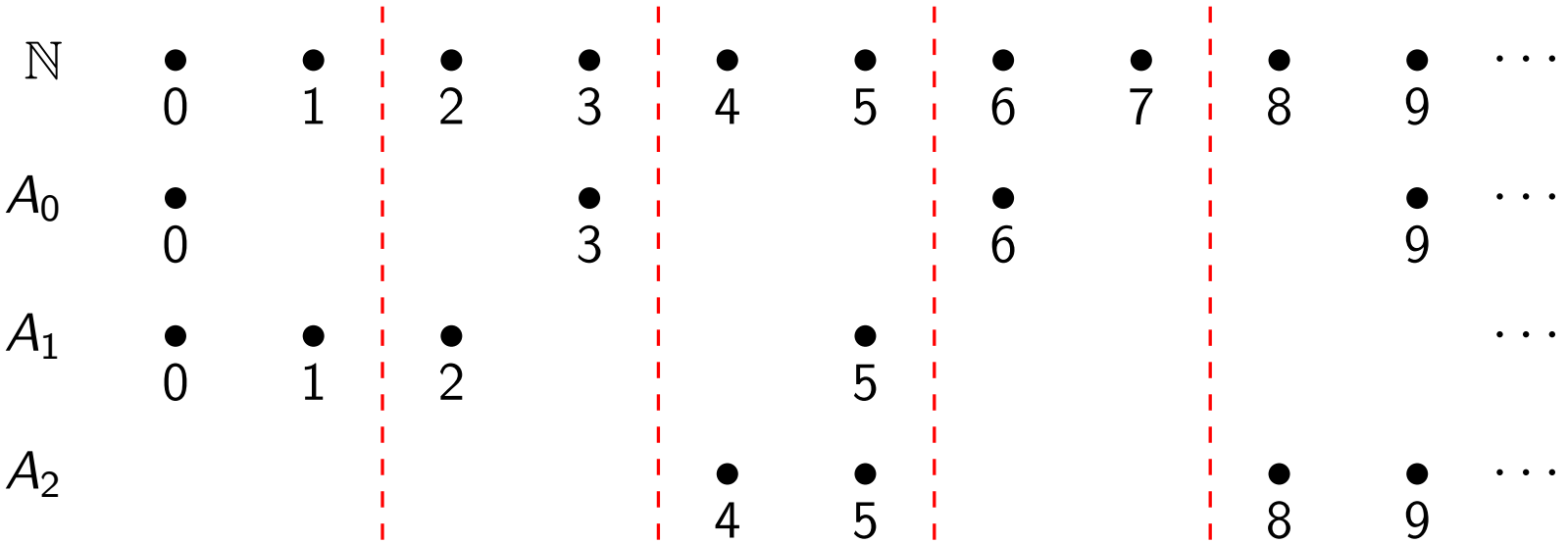
$$f \restriction [a, A] : [a, A] \to [\overline{f}(a), f(A)]$$

is (cofinal continuous) monotone quasi-open.

We just show that f is monotone quasi-open. The same argument holds on any Ellentuck neighbourhood.

Let $g := \gcd(D)$. We first consider the case where $D = g\mathbb{N}$.

D-Ellentuck space $([\mathbb{N}]^{\infty,D}, \subseteq^D, r)$, $D = \{0, 2, 4, 6, \dots\}$



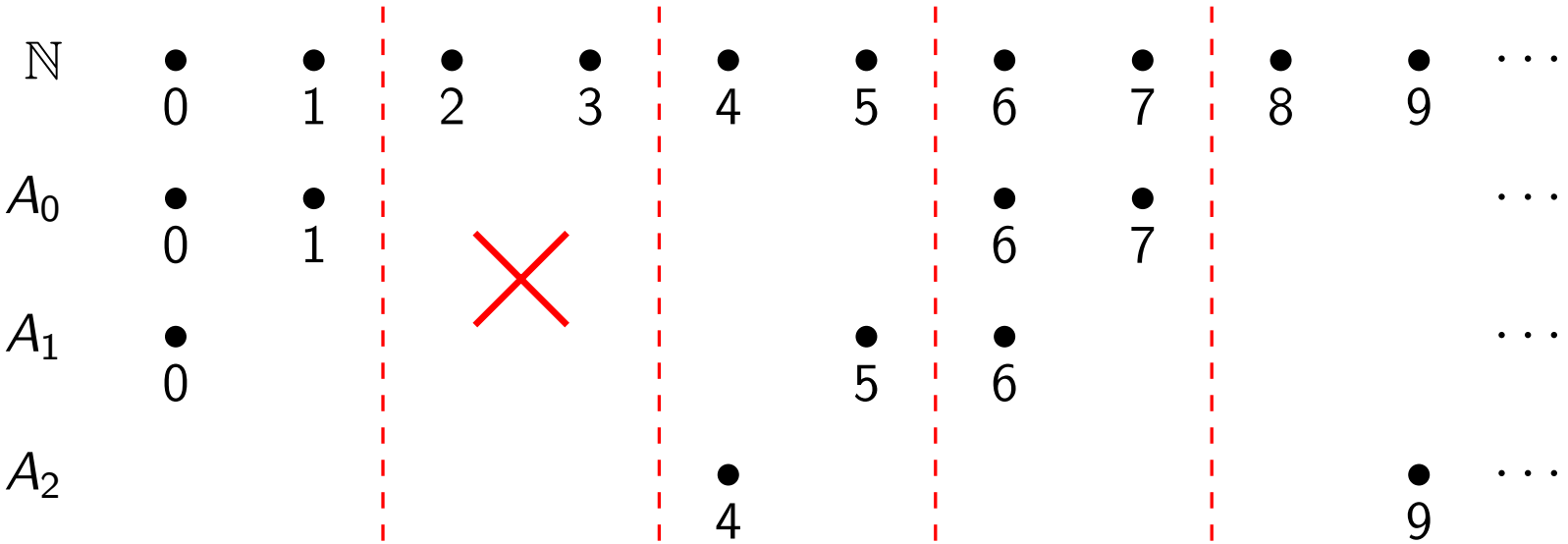
$$f(A) := \{k < \omega : gk \in A\}.$$

(i.e. all partitions that contain some element of A)

Then $[f(A)] = f\text{“}[A]$ for all A

Let $g := \gcd(D)$. What if $D \subsetneq g\mathbb{N}$ (say $D \setminus N = g\mathbb{N} \setminus N$)?

D-Ellentuck space $([\mathbb{N}]^{\infty,D}, \subseteq^D, r)$, $D = \{0, 4, 6, 8, \dots\}$



$$f(A) := \{k < \omega : gk \in A\}$$

(i.e. all partitions that contain some element of A)

Then $[f(A) \setminus N] \subseteq f\text{“}[A]$ for all A

Back to topological Ramsey spaces. Recall this theorem:

Theorem

If $(\mathcal{R}, \leq, r)$ is depth-preserving and $D_{a,A} = \mathbb{N}$ for all $A \in \mathcal{R}$ and $a \in \mathcal{AR}{\restriction}A$, then the map:

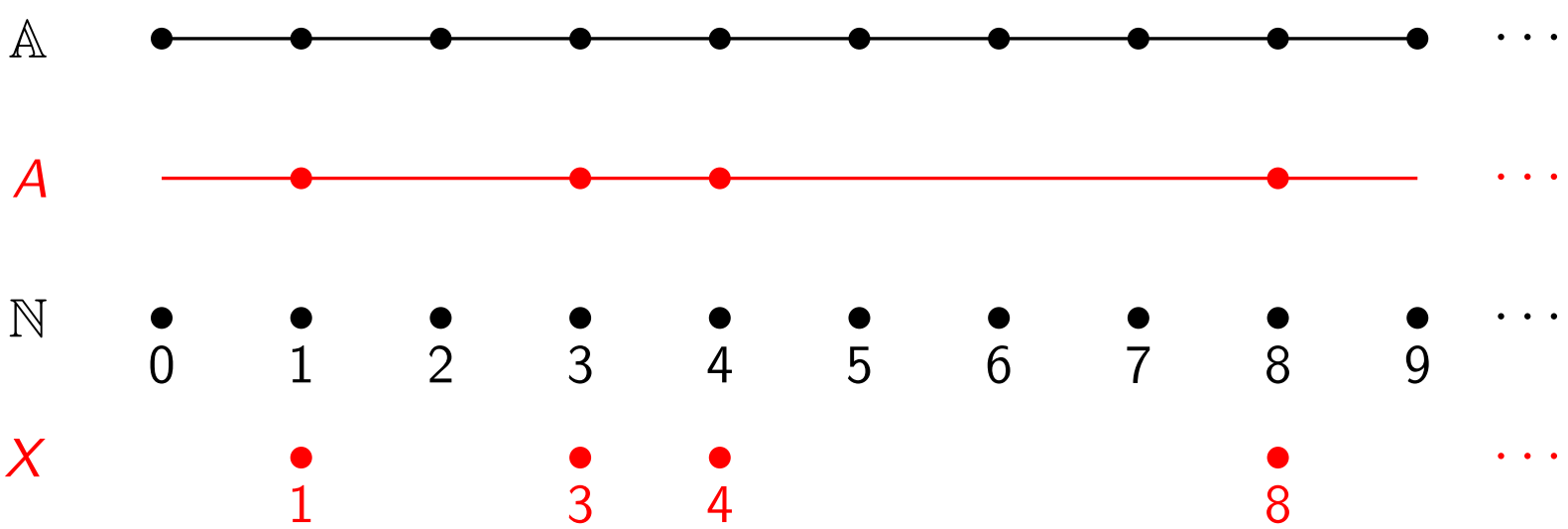
$$f(A) := \{\mathrm{depth}(r_n(A)) - 1 : 0 < n < \omega\}$$

satisfies the following:

1. 1.1 *If $a \sqsubseteq b$, then $\overline{f}(a) \sqsubseteq \overline{f}(b)$.*
   1.2 *If $a \leq_{\mathrm{fin}} b$, then $\overline{f}(a) \subseteq \overline{f}(b)$.*
2. *For all $a \in \mathcal{AR}$, $\mathrm{lh}(\overline{f}(a)) = \mathrm{lh}(a)$.*
3. *$f(\mathbb{A}) = \mathbb{N}$.*
4. *For all $A \in \mathcal{R}$ and $a \in \mathcal{AR}{\restriction}A$, $f\text{“}[a, A] = [\overline{f}(a), f(A)]$.*

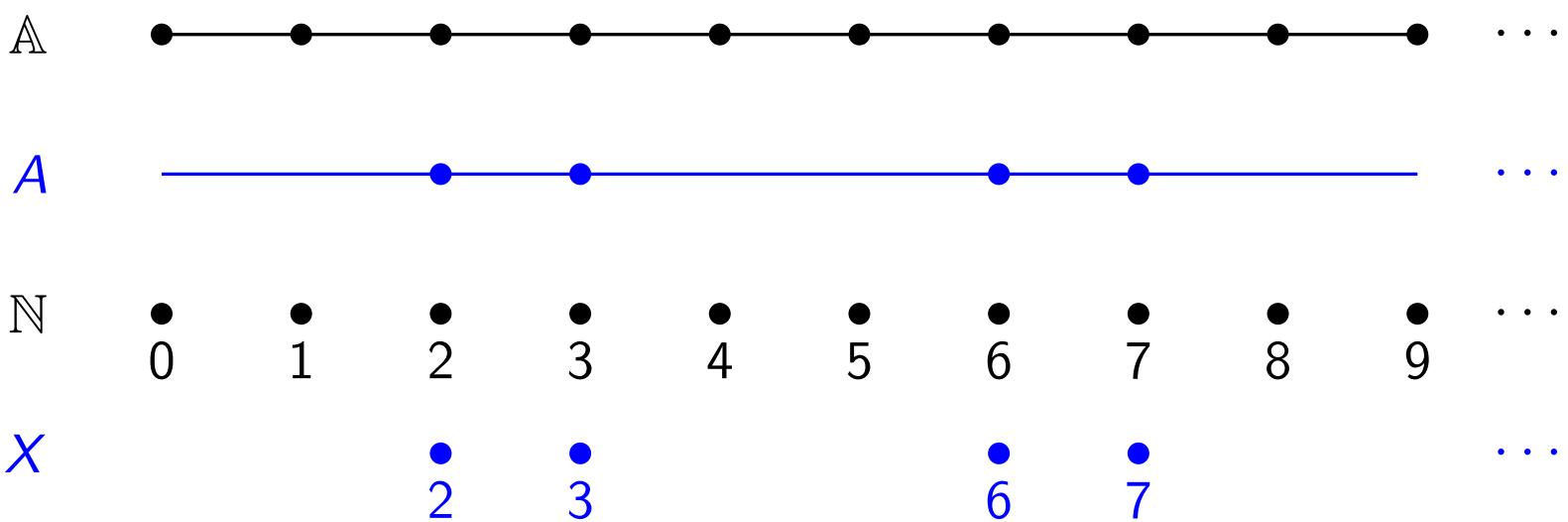
We proved it by "picking the next element one-by-one".

Projection $(\mathcal{R}, \leq, r) \to ([\mathbb{N}]^\infty, \subseteq, r)$



What if $D(\mathcal{R}) \neq \mathbb{N}$, say $D(\mathcal{R}) = \{0, 2, 4, 6, \dots\}$?

Projection $(\mathcal{R}, \leq, r) \to ([\mathbb{N}]^{\infty, D(\mathcal{R})}, \subseteq^{D(\mathcal{R})}, r)$



Therefore, by the same argument:

Theorem

If $(\mathcal{R}, \leq, r)$ is depth-preserving depth-uniform, with $D := D(\mathcal{R})$, then the map $f : \mathcal{R} \to [\mathbb{N}]^{\infty, D}$:

$$f(A) := \{\mathrm{depth}(r_n(A)) - 1 : 0 < n < \omega\}$$

satisfies the following:

1. 1.1 *If $a \sqsubseteq b$, then $\overline{f}(a) \sqsubseteq \overline{f}(b)$.*
   1.2 *If $a \leq_{\mathrm{fin}} b$, then $\overline{f}(a) \subseteq \overline{f}(b)$.*
2. *For all $a \in \mathcal{AR}$, $\mathrm{lh}(\overline{f}(a)) = \mathrm{lh}(a)$.*
3. *$f(\mathbb{A}) = \mathbb{N}$.*
4. *For all $A \in \mathcal{R}$ and $a \in \mathcal{AR} \restriction A$, $f\text{“}[a, A] = [\overline{f}(a), f(A)]$.*

Combined with the theorem that D-Ellentuck space projects into Ellentuck space for $D \neq \{0\}$, we get:

Theorem

If $(\mathcal{R}, \leq, r)$ is depth-preserving depth-uniform with $D(\mathcal{R}) \neq \{0\}$, then there is some $f : \mathcal{R} \to [\mathbb{N}]^\infty$ such that for all $A \in \mathcal{R}$ and $a \in \mathcal{AR}{\restriction}A$, the restricted map:

$$f{\restriction}[a, A] : [a, A] \to [\overline{f}(a), f(A)]$$

is (cofinal continuous) monotone quasi-open.

When $D = \{0\}$

What does it mean when $(\mathcal{R}, \leq, r)$ satisfies $D(\mathcal{R}) = \{0\}$?

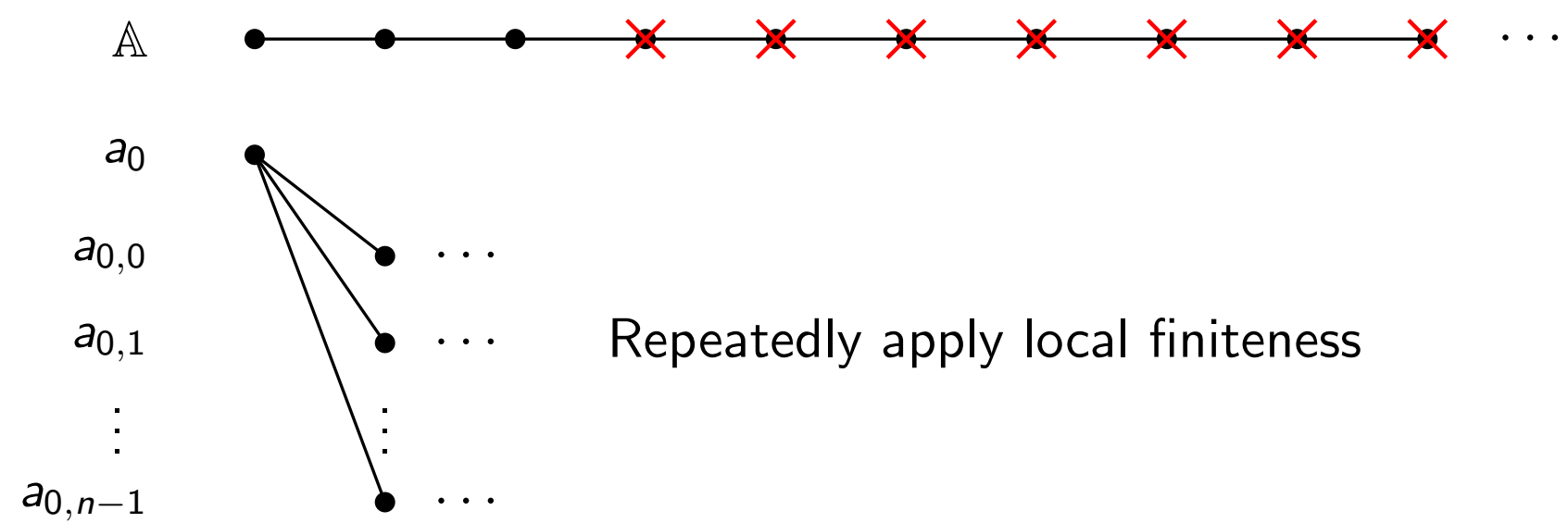
Lemma

If $D(\mathcal{R}) = \{0\}$, then for each $n < \omega$, there are only finitely many sequences of length $\leq n$.

Lemma

If $D(\mathcal{R}) = \{0\}$, then for each $n < \omega$, there are only finitely many sequences of length $\leq n$.

Depth-uniform $(\mathcal{R}, \leq, r)$, with $D(\mathcal{R}) = \{0\}$



$\{a_{0,0}, \ldots, a_{0,n-1}\} = \{b : a_0 \sqsubseteq b \text{ and } b \leq_{\text{fin}} r_2(\mathbb{A})\}$ **(Locally finite)**

This means that $(\mathcal{AR}, \sqsubseteq)$ is a finitely-branching tree. To get a $\leq$-minimal element, repeatedly take the $\leq_{\mathrm{fin}}$-least one-point extension. This works in any cone $[A]$ of $\mathcal{R}$.

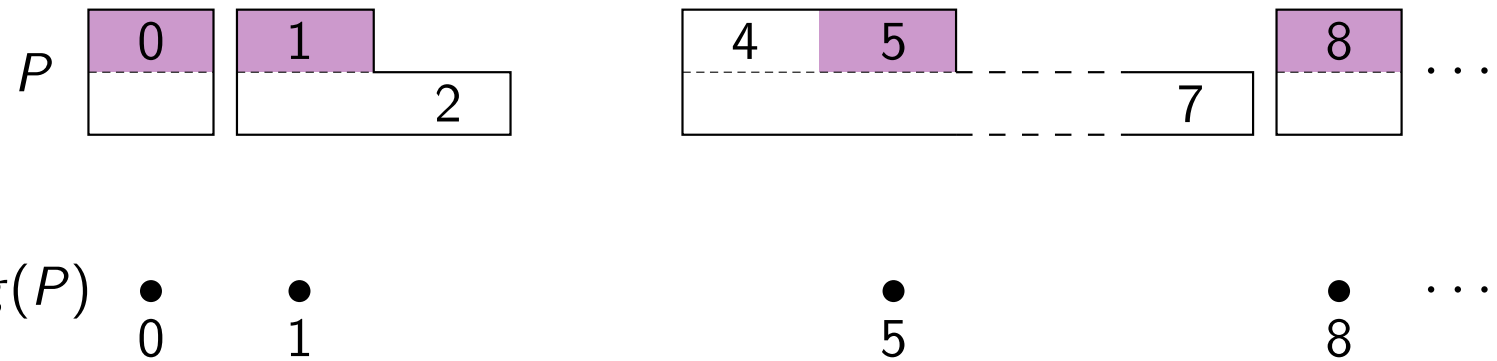
Theorem

If $D(\mathcal{R}) = \{0\}$, then $(\mathcal{R}, \leq)$ contains a dense set of atoms (i.e $\leq$-minimal elements).

Depth-preserving...?

We can't remove the depth-uniform hypothesis. But what about the depth-preserving hypothesis?

Projection $(\mathbf{FIN}_2^{[\infty]}, \leq, r) \to ([\mathbb{N}]^\infty, \subseteq, r)$



Proposition

The map $g := \max^* \circ T : \mathbf{FIN}_2^{[\infty]} \to [\mathbb{N}]^\infty$ *is cofinal continuous monotone, and* $g``[a, P] = [\overline{g}(a), g(P)]$ *for all* a, P.

Problem

Let $(\mathcal{R}, \leq, r)$ be depth-uniform. Does $(\mathcal{R}, \leq, r)$ project into $([\mathbb{N}]^\infty, \subseteq, r)$? What if $D(\mathcal{R}) = \mathbb{N}$?

The key difficulty lies in the lack of a natural choice of a projection for a topological Ramsey space apart from (some variant of) the depth map.

Thanks for listening!

- **Depth-preserving.** If $a \in \mathcal{AR}\restriction B$ and $B \leq A$:

$$\mathrm{depth}_A(a) = \mathrm{depth}_A(r_{\mathrm{depth}_B(a)}(A)).$$

- **Depth-uniform.** If $a \in \mathcal{AR}\restriction A$, let:

$$D_{a,A} := \{\mathrm{depth}_A(b) - \mathrm{depth}_A(a) - 1 : b \in r_{\mathrm{lh}(a)+1}[a, A]\}.$$

  Then $D_{a,A} = D_{b,B}$ for all a, A, b, B. Call this set $D(\mathcal{R})$.
- **Dichtomy theorem.**
  - If $D(\mathcal{R}) \neq \{0\}$, then there is some $f : \mathcal{R} \to [\mathbb{N}]^\infty$ that is cofinal continuous monotone quasi-open on every Ellentuck neighbourhood.
  - If $D(\mathcal{R}) = \{0\}$, then $(\mathcal{R}, \leq)$ has a dense set of atoms.