Weak A2 spaces, the Kastanas game and strategically Ramsey sets

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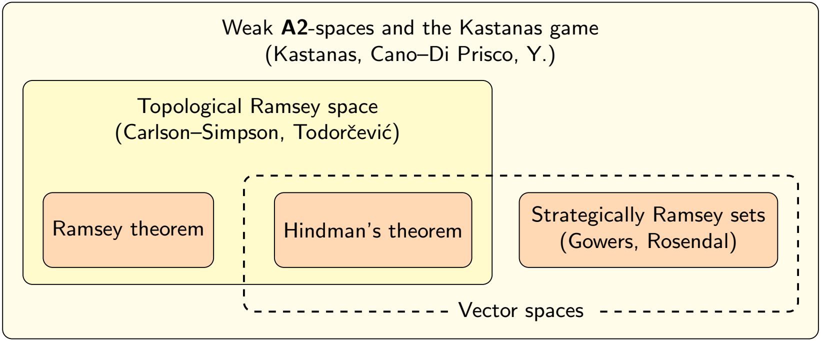

Weak **A2**-spaces and the Kastanas game
(Kastanas, Cano–Di Prisco, Y.)
Topological Ramsey space
(Carlson–Simpson, Todorčević)
Ramsey theorem
Hindman's theorem
Strategically Ramsey sets
(Gowers, Rosendal)
Vector spaces

(Finite-dimensional) Ramsey Theory

In general, Ramsey theory addresses the following types of questions:

Let X be a set or structure, and let $Y_0, \ldots, Y_{n-1}$ be a partition of X. Does there exist some $i < n$ such that Y_i contains some substructure of interest?

Example.

1. X is infinite.
2. Structure = Infinite set.

Fact (Pigeonhole principle)

Let X be an infinite set. If $Y_0, \ldots, Y_{n-1}$ is a partition of X, then there exists some $i < n$ such that Y_i is infinite.

Example.

1. $X = [\mathbb{N}]^n =$ Set of n-element subsets of $\mathbb{N}$.
2. Structure $= [H]^n$ for some infinite $H \subseteq \mathbb{N}$.

Theorem (Ramsey, 1929)

If $Y_0, \ldots, Y_{n-1}$ is a partition of $[\mathbb{N}]^n$, then there exists some $i < n$ and an infinite $H \subseteq \mathbb{N}$ such that $[H]^n \subseteq Y_i$.

Example. Let $\mathbb{F}$ be a countable (possibly finite) field. Let E be a vector space over $\mathbb{F}$ of dimension $\aleph_0$, with Hamel basis $(e_n)_{n=0}^{\infty}$. Given a vector $x \in E$, we may write

$$x = \sum_{n=0}^{\infty} \lambda_n(x) e_n,$$

where only finitely many λ_n's are non-zero. We may then write:

$$\operatorname{supp}(x) := \{n \in \mathbb{N} : \lambda_n(x) \neq 0\}.$$

Notation

Given two non-zero vectors x, y we write:

$$x < y \iff \max(\operatorname{supp}(x)) < \min(\operatorname{supp}(y)).$$

Definition

An *infinite block sequence* is a $<$-increasing sequence of non-zero vectors of E.

Definition

An infinite-dimensional subspace $V \subseteq E$ is a *block subspace* if $V = \operatorname{span}\{x_n : n \in \mathbb{N}\}$ for some infinite block sequence $(x_n)_{n=0}^{\infty}$. Note that $\{x_n\}_{n=0}^{\infty}$ is a basis of V that is unique up to scalar multiplication of each x_n.

Fact

Every infinite-dimensional subspace of E contains an infinite-dimensional block subspace.

Now consider the following setting:

1. $X = E \setminus \{0\}$, the set of non-zero vectors.
2. Structure = Infinite-dimensional block subspace (without 0).

Does the Ramsey theorem hold for this variant?

Theorem (Hindman, 1975)

Suppose that $|\mathbb{F}| = 2$. If $Y_0, \ldots, Y_{n-1}$ is a partition of $E \setminus \{0\}$, then there exists some $i < n$ and some infinite-dimensional block subspace V such that $V \setminus \{0\} \subseteq Y_i$.

This theorem fails if $|\mathbb{F}| > 2$. We define the set Y as:

$$\begin{aligned} Y &= \{x \in E \setminus \{0\} : x = e_n + y \text{ for some } e_n < y\}, \\ &= \left\{ x \in E \setminus \{0\} : \begin{array}{c} x = e_{n_0} + \lambda_{n_1} e_{n_1} + \cdots + \lambda_{n_k} e_{n_k} \\ \text{and } n_0 < \cdots < n_k \end{array} \right\}. \end{aligned}$$

Then Y, Y^c partition $E \setminus \{0\}$, but neither Y nor Y^c contains $V \setminus \{0\}$ for any infinite-dimensional block subspace V.

Example

Let $A = (e_0 + e_1, e_2 + e_3, \dots)$. Then $e_0 + e_1 \in Y$, but $2e_0 + 2e_1 \in Y^c$, so $\operatorname{span}(A) \setminus \{0\}$ is not a subset of either Y or Y^c.

Infinite-dimensional Ramsey theory

For infinite-dimensional Ramsey theory, we partition $X^{\mathbb{N}}$ or a closed subset $\mathcal{R}$ of $X^{\mathbb{N}}$. Here we equip X with the discrete topology, and $X^{\mathbb{N}}$ with the product topology.

More precisely:

Let X be a set or structure. Let $\mathcal{X}_0, \ldots, \mathcal{X}_{n-1}$ be a partition of $\mathcal{R}$, a closed subset of $X^{\mathbb{N}}$. Does there exist some $i < n$ such that $\mathcal{X}_i$ contains some substructure of interest?

Example.

1. $X = \mathbb{N}$.
2. $\mathcal{R} = [\mathbb{N}]^\infty$, which may be identified with the set of strictly increasing sequences in $\mathbb{N}^\mathbb{N}$.
3. Structure = *Ellentuck neighbourhood* of infinite subset.

Let $[\mathbb{N}]^{<\infty}\restriction A := \{a \in [\mathbb{N}]^{<\infty} : a \subseteq A\}$.

Notation

Given $A \in [\mathbb{N}]^{\infty}$ and $a \in [\mathbb{N}]^{<\infty}\restriction A$, we write:

$$[a, A] := \{B \in [\mathbb{N}]^{\infty} : a \sqsubseteq B \text{ and } B \subseteq A\}$$

where $a \sqsubseteq B$ means that $B \cap (\max(a) + 1) = a$.

Each $[a, A]$ is also called an *Ellentuck neighbourhood*.

Definition

A set $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is *Ramsey* if for all $A \in [\mathbb{N}]^\infty$ and $a \in [\mathbb{N}]^{<\infty} \restriction A$, there exists some $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \subseteq \mathcal{X}^c$.

Fact

Assuming the Axiom of Choice, there is a subset $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ that is not Ramsey.

Recall that a subset $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is *analytic* if it is the continuous image of a Borel set in some Polish space. Note that every Borel set is analytic.

Theorem (Silver, 1970)

If $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is analytic, then $\mathcal{X}$ is Ramsey.

Example. Let E be a vector space over a countable field $\mathbb{F}$ of dimension $\aleph_0$, with Hamel basis $(e_n)_{n=0}^{\infty}$.

1. $X = E$.
2. $\mathcal{R} = E^{[\infty]}$, the set of all infinite block sequences of vectors ($\Leftrightarrow$ infinite-dimensional block subspaces).
3. Structure = Ellentuck neighbourhood of infinite-dimensional block subspaces.

Notation

If $A = (x_n)_{n=0}^{\infty}$ and $B = (y_n)_{n=0}^{\infty}$ are two elements of $E^{[\infty]}$, then we write:

$$B \leq A \iff \mathrm{span}(B) \subseteq \mathrm{span}(A).$$

Notation

Given $A = (x_n)_{n=0}^{\infty} \in E^{[\infty]}$, let $E^{[<\infty]}\restriction A$ be the set of finite block subspaces of A. In other words, the set of $(y_m)_{m<N}$ such that $\operatorname{span}\{y_0, \dots, y_{N-1}\} \subseteq \operatorname{span}(A)$.

Notation

Given $A \in E^{[\infty]}$ and $a \in E^{[<\infty]}\restriction A$, we write:

$$[a, A] := \{B \in E^{[\infty]} : a \sqsubseteq B \text{ and } B \leq A\}$$

where $a \sqsubseteq B$ means that a is an initial segment of B.

Definition

A set $\mathcal{X} \subseteq E^{[\infty]}$ is *Ramsey* if for all $A \in E^{[\infty]}$ and $a \in E^{[<\infty]} \upharpoonright A$, there exists some $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \subseteq \mathcal{X}^c$.

Fact

Assuming the Axiom of Choice, there is a subset $\mathcal{X} \subseteq E^{[\infty]}$ that is not Ramsey.

Theorem (Hindman, 1975)

Suppose that $|\mathbb{F}| = 2$. If $\mathcal{X} \subseteq E^{[\infty]}$ is analytic, then it is Ramsey.

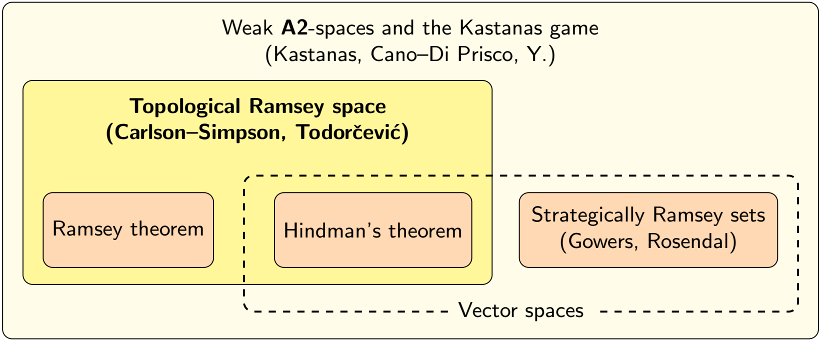
Again, this theorem fails for $|\mathbb{F}| > 2$ — there exists a clopen subset $\mathcal{X}$ of $E^{[\infty]}$ which is not Ramsey.

We observe some patterns:

1. $\mathcal{R}$ is a set of infinite increasing sequences under some partial order $<$.
2. Using either $\subseteq$ or $\leq$, we defined the Ellentuck neighbourhood $[a, A]$, and the notion of a Ramsey subset $\mathcal{X} \subseteq \mathcal{R}$.

Question. Can this pattern be captured and made into an abstract framework?

Topological Ramsey theory



Framework of topological Ramsey spaces

(Originally Carlson–Simpson (1990), modified by Todorčević (2010))

Object	**Represents**	**Ex. #1**	**Ex. #2**
$\mathcal{R}$	Infinite increasing seqs	$[\mathbb{N}]^\infty$	$E^{[\infty]}$
$\leq$	Quasi-order on $\mathcal{R}$	Subset $\subseteq$	Subspace $\leq$
$\mathcal{AR}$	Finite increasing seqs	$[\mathbb{N}]^{<\infty}$	$E^{[<\infty]}$
$r_n : \mathcal{R} \to \mathcal{AR}$ for $n \in \mathbb{N}$	First n elements of the sequences	$r_n(A) = \{x_0, \ldots, x_{n-1}\}$	$r_n(A) = (x_0, \ldots, x_{n-1})$

Represented by a triple $(\mathcal{R}, \leq, r)$, that satisfies four axioms **A1–A4** and a closured property.

Notation

Given a triple $(\mathcal{R}, \leq, r)$, for $A \in \mathcal{R}$ and $a \in \mathcal{AR}$:

$$a \sqsubseteq A \iff a = r_n(A) \text{ for some } n \in \mathbb{N}.$$

Notation

If $A \in \mathcal{R}$ and $a \in \mathcal{AR}$,

$$[a, A] := \{B \in \mathcal{R} : a \sqsubseteq B \text{ and } B \leq A\}.$$

Axiom (A1, Sequencing)

(1) $r_0(A) = \emptyset$ for all $A \in \mathcal{R}$.

(2) $A \neq B$ implies $r_n(A) \neq r_n(B)$ for some n.

(3) $r_n(A) = r_m(B)$ implies $n = m$ and $r_k(A) = r_k(B)$ for all $k < n$.

Axiom (**A2**, Finitisation)

There is a quasi-ordering $\leq_{\text{fin}}$ on $\mathcal{AR}$ such that:

(1) $\{b \in \mathcal{AR} : b \leq_{\text{fin}} a\}$ is finite for all $a \in \mathcal{AR}$.

(2) $A \leq B$ iff $\forall n \exists m [r_n(A) \leq_{\text{fin}} r_m(B)]$.

(3) $\forall a, b \in \mathcal{AR}[a \sqsubseteq b \wedge b \leq_{\text{fin}} c \to \exists d \sqsubseteq c[a \leq_{\text{fin}} d]]$.

1. $([\mathbb{N}]^{\infty}, \subseteq, r)$ satisfies **A2**, as for all $a \in [\mathbb{N}]^{<\infty}$, $\{b : b \subseteq a\}$ is finite.
2. $(E^{[\infty]}, \leq, r)$ satisfies **A2** if $|\mathbb{F}| < \infty$, as if $a = (x_i)_{i<n} \in E^{[<\infty]}$, then there are finitely many subspaces of a.
3. $(E^{[\infty]}, \leq, r)$ does not satisfy **A2** if $|\mathbb{F}| = \infty$, as if $a = (x_0, x_1)$, then $\text{span}\{\lambda x_0 + x_1\}$ is a (block) subspace for any $\lambda \in \mathbb{F}$.

Axiom (A3, Amalgamation)

The depth function defined by, for $B \in \mathcal{R}$ and $a \in \mathcal{AR}$:

$$\mathrm{depth}_B(a) := \begin{cases} \min\{n \in \mathbb{N} : a \leq_{\mathrm{fin}} r_n(B)\}, & \text{if such } n \text{ exists} \\ \infty, & \text{otherwise} \end{cases}$$

satisfies the following:

(1) If $\mathrm{depth}_B(a) < \infty$, then for all $A \in [\mathrm{depth}_B(a), B]$, $[a, A] \neq \emptyset$.

(2) $A \leq B$ and $[a, A] \neq \emptyset$ imply that there exists $A' \in [\mathrm{depth}_B(a), B]$ such that $\emptyset \neq [a, A'] \subseteq [a, A]$.

We let $\mathcal{AR}_n$ be the image of the map $r_n(-)$, i.e. the set of all finite approximations of length n.

Axiom (**A4**, Pigeonhole)

If $\mathrm{depth}_B(a) < \infty$ and if $\mathcal{O} \subseteq \mathcal{AR}_{\mathrm{lh}(a)+1}$, then there exists $A \in [\mathrm{depth}_B(a), B]$ such that $r_{\mathrm{lh}(a)+1}[a, A] \subseteq \mathcal{O}$ or $r_{\mathrm{lh}(a)+1}[a, A] \subseteq \mathcal{O}^c$.

Pigeonhole should be viewed as a statement about the finite-dimensional Ramsey theorem for $\mathcal{R}$.

1. $([\mathbb{N}]^\infty, \subseteq, r)$ satisfies **A4**, due to the pigeonhole principle.
2. $(E^{[\infty]}, \leq, r)$ satisfies **A4** if $|\mathbb{F}| = 2$, due to Hindman's theorem.
3. $(E^{[\infty]}, \leq, r)$ does not satisfy **A4** if $|\mathbb{F}| > 2$, as Hindman's theorem fails for $|\mathbb{F}| > 2$.

Observe that every $A \in \mathcal{R}$ can be uniquely identified with the sequence $(r_n(A))_{n=0}^{\infty}$, an element of $\mathcal{AR}^{\mathbb{N}}$. Therefore, we may identify $\mathcal{R}$ with a subset of $\mathcal{AR}^{\mathbb{N}}$.

Recall that in infinite-dimensional Ramsey theory, we require $\mathcal{R}$ to be a closed subset of $X^{\mathbb{N}}$. We make a similar requirement here.

Definition

$(\mathcal{R}, \leq, r)$ is a *closed triple* if for every $\sqsubseteq$-increasing sequence $(a_n)_{n=0}^{\infty}$ of elements in $\mathcal{AR}$ such that $\mathrm{lh}(a_n) = n$, there exists some $A \in \mathcal{R}$ such that $r_n(A) = a_n$ for all n.

In other words, $\mathcal{R}$ is a metrically closed subset of $\mathcal{AR}^{\mathbb{N}}$.

Definition

A *topological Ramsey space* is a closed triple $(\mathcal{R}, \leq, r)$ satisfying **A1–A4**.

Definition

Let $(\mathcal{R}, \leq, r)$ be a topological Ramsey space. A set $\mathcal{X} \subseteq \mathcal{R}$ is *Ramsey* if for all $A \in \mathcal{R}$ and $a \in \mathcal{AR}\restriction A$, there exists some $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \subseteq \mathcal{X}^c$.

Theorem (Todorčević, 2010)

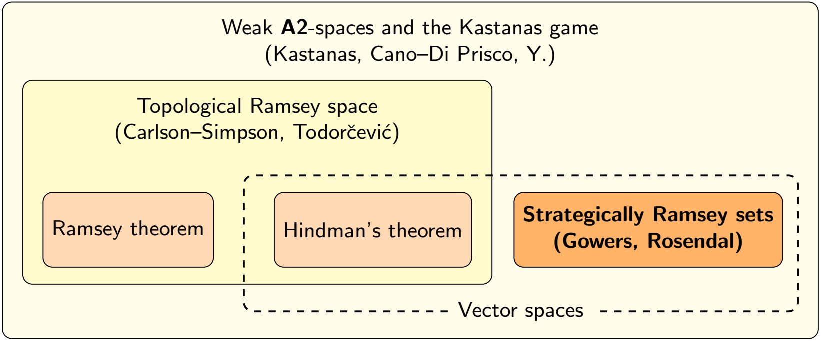
Let $(\mathcal{R}, \leq, r)$ be a topological Ramsey space such that $\mathcal{AR}$ is countable. Then every analytic set is Ramsey.

Examples

$(\mathcal{R}, \leq, r)$	Closed triple	A1	A2	A3	A4	Topological Ramsey space
$([\mathbb{N}]^\infty, \subseteq, r)$	✓	✓	✓	✓	✓	✓
$(E^{[\infty]}, \leq, r)$ $\lvert\mathbb{F}\rvert = 2$	✓	✓	✓	✓	✓	✓
$(E^{[\infty]}, \leq, r)$ $2 < \lvert\mathbb{F}\rvert < \infty$	✓	✓	✓	✓		
$(E^{[\infty]}, \leq, r)$ $\lvert\mathbb{F}\rvert = \infty$	✓	✓		✓		

Question. Is there an alternative framework of Ramsey theory for countable vector spaces over $\mathbb{F}$ with $|\mathbb{F}| > 2$?

Strategically Ramsey sets



Banach space theory

It turns out that there is a deep connection between Banach space theory and the Ramsey theory for (countable) vector spaces.

Question (Banach, 1932)

Let X be an infinite-dimensional Banach space, and suppose that X is *homogeneous* (i.e. X is isomorphic to every one of its infinite-dimensional closed subspaces). Must X be isomorphic to ℓ^2?

This was answered in the positive by Gowers (2002), and the proof is heavily Ramsey-theoretic and uses ideas from infinite games.

We fix some countable vector space E. Let $A \in E^{[\infty]}$ be an infinite block sequence, and let $a \in E^{[<\infty]}\restriction A$.

Definition (Gowers, 2002)

The *Gowers game* played below $[a, A]$, denoted as $G[a, A]$, is the following game:

$$\begin{array}{c|ccccc} \textbf{I} & A_0 \leq A & & A_1 \leq A & & \cdots \\ \hline \textbf{II} & & x_0 \in \langle A_0 \rangle & & x_1 \in \langle A_1 \rangle & \cdots \end{array}$$

where $\langle A_i \rangle$ is the span of the elements in the sequence A_i. The outcome of this game is the sequence $a^\frown (x_k)_{k=0}^{\infty} \in E^{[\infty]}$.

Gowers' surprising application of Ramsey theory to Banach space theory motivated other similar games played on vector spaces.

Definition (Rosendal, 2010)

The *asymptotic game* played below $[a, A]$, denoted as $F[a, A]$, is the following game:

$$
\begin{array}{c|cccc}
\mathbf{I} & n_0 & & n_1 & & \cdots \\
\hline
\mathbf{II} & & x_0 \in \langle A \rangle & & x_1 \in \langle A \rangle & \cdots \\
 & & \min(\operatorname{supp}(x_0)) > n_0 & & \min(\operatorname{supp}(x_1)) > n_1 &
\end{array}
$$

The outcome of this game is the sequence $a^\frown(x_k)_{k=0}^{\infty} \in E^{[\infty]}$.

Definition

We say that **I** (similarly **II**) has a strategy in a game to *reach* $\mathcal{X} \subseteq E^{[\infty]}$ if they have a strategy in the game to ensure the outcome is in $\mathcal{X}$.

Definition (Rosendal, 2010)

A subset $\mathcal{X} \subseteq E^{[\infty]}$ is *strategically Ramsey* if for all $A \in E^{[\infty]}$ and $a \in E^{[<\infty]} \restriction A$, there exists some $B \in [a, A]$ such that one of the following holds:

1. **I** has a strategy in $F[a, B]$ to reach $\mathcal{X}^c$.
2. **II** has a strategy in $G[a, B]$ to reach $\mathcal{X}$.

Definition

We say that **I** (similarly **II**) has a strategy in a game to *reach* $\mathcal{X} \subseteq \mathcal{R}$ if they have a strategy in the game to ensure the outcome is in $\mathcal{X}$.

Definition (Rosendal, 2010)

A subset $\mathcal{X} \subseteq E^{[\infty]}$ is *strategically Ramsey* if for all $A \in E^{[\infty]}$ and $a \in E^{[<\infty]} \restriction A$, there exists some $B \in [a, A]$ such that one of the following holds:

1. **I** has a strategy in $F[a, B]$ to reach $\mathcal{X}^c$.
   (Definition of Ramsey: $[a, B] \subseteq \mathcal{X}^c$.)
2. **II** has a strategy in $G[a, B]$ to reach $\mathcal{X}$.
   (Definition of Ramsey: $[a, B] \subseteq \mathcal{X}$.)

Theorem (Rosendal, 2010)

Every analytic subset of $E^{[\infty]}$ is strategically Ramsey.

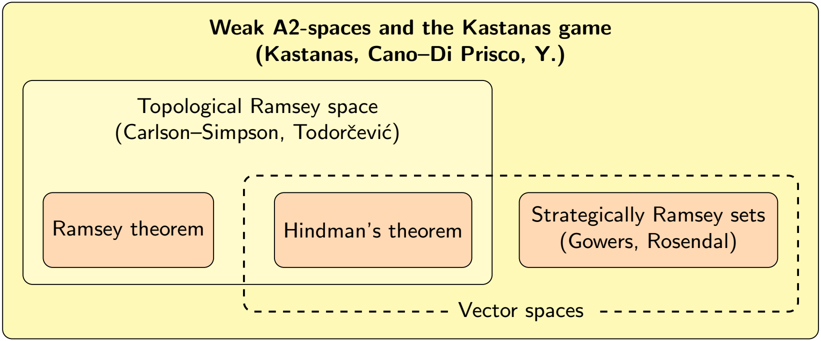
Note that there is a coanalytic non-strategically Ramsey set in Gödel's constructible universe L.

wA2-spaces and the Kastanas game

$(\mathcal{R}, \leq, r)$	**A2**	**A4**	Topological Ramsey space	Strategically Ramsey sets
$([\mathbb{N}]^\infty, \subseteq, r)$	✓	✓	✓	
$(E^{[\infty]}, \leq, r)$ $\lvert\mathbb{F}\rvert = 2$	✓	✓	✓	✓
$(E^{[\infty]}, \leq, r)$ $2 < \lvert\mathbb{F}\rvert < \infty$	✓			✓
$(E^{[\infty]}, \leq, r)$ $\lvert\mathbb{F}\rvert = \infty$				✓

Question. Is there a framework unifying topological Ramsey theory and the theory of strategically Ramsey sets?

Weak **A2** spaces



We shall unify the two frameworks by doing two things.

1. Weakening the axioms **A1–A4** to include countable vector spaces with $|\mathbb{F}| > 2$.
2. Finding a "common game" that connects Ramsey sets to strategically Ramsey sets.

1. Weakening axioms. The framework of topological Ramsey spaces does not apply to countable vector spaces in full generality for two reasons.

- **A2** fails if $|\mathbb{F}| = \infty$ (a finite-dimensional subspace may have infinitely many further subspaces).
- **A4** fails if $|\mathbb{F}| > 2$ (Hindman's theorem fails for $|\mathbb{F}| > 2$).

Axiom (**wA2**, Weak Finitisation)

There is a quasi-ordering $\leq_{\mathrm{fin}}$ on $\mathcal{AR}$ such that:

(w1) $\{b \in \mathcal{AR} : b \leq_{\mathrm{fin}} a\}$ is countable for all $a \in \mathcal{AR}$.

(2) $A \leq B$ iff $\forall n \exists m [r_n(A) \leq_{\mathrm{fin}} r_m(B)]$.

(3) $\forall a, b \in \mathcal{AR}[a \sqsubseteq b \wedge b \leq_{\mathrm{fin}} c \rightarrow \exists d \sqsubseteq c[a \leq_{\mathrm{fin}} d]]$.

Definition

A triple $(\mathcal{R}, \leq, r)$ is a *weak* **A2** *space*, or just **wA2**-*space*, if it is a closed triple satisfying **A1**, **wA2**, **A3**.

All examples we saw are **wA2**-spaces:

$(\mathcal{R}, \leq, r)$	**Closed triple**	**A1**	A2	**wA2**	**A3**	A4
$([\mathbb{N}]^\infty, \subseteq, r)$	✓	✓	✓	✓	✓	✓
$(E^{[\infty]}, \leq, r)$ $\lvert\mathbb{F}\rvert = 2$	✓	✓	✓	✓	✓	✓
$(E^{[\infty]}, \leq, r)$ $2 < \lvert\mathbb{F}\rvert < \infty$	✓	✓	✓	✓	✓	
$(E^{[\infty]}, \leq, r)$ $\lvert\mathbb{F}\rvert = \infty$	✓	✓		✓	✓	

2. Finding a "common game". One may consider generalising the notion of strategically Ramsey sets to all **wA2**-spaces, but this is not straightforward — recall the asymptotic game $F[a, A]$:

I	n_0		n_1		$\cdots$
II		$x_0 \in \langle A \rangle$ $\min(\text{supp}(x_0)) > n_0$		$x_1 \in \langle A \rangle$ $\min(\text{supp}(x_1)) > n_1$	$\cdots$

For **wA2**-spaces in general, there is not necessarily a notion of *support*.

It turns out that there is another game that is relevant.

Definition (Kastanas, 1983)

Let $A \in [\mathbb{N}]^\infty$ and $a \in [\mathbb{N}]^{<\infty} \restriction A$. The *Kastanas game* played below $[a, A]$, denoted as $K[a, A]$, is:

I	$A_0 \subseteq A$		$A_1 \subseteq B_0$	$\cdots$
II		$n_0 \in A_0$ $n_0 > \max(a)$ $B_0 \subseteq A_0$		$n_1 \in A_1$ $n_1 > n_0$ $B_1 \subseteq A_1$

The outcome of this game is $a \cup \{n_0, n_1, \ldots\}$.

Definition

A set $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is *Kastanas Ramsey* if for all $A \in [\mathbb{N}]^\infty$ and $a \in [\mathbb{N}]^{<\infty} \restriction A$, there exists some $B \in [a, A]$ such that one of the following holds:

1. **I** has a strategy in $K[a, B]$ to reach $\mathcal{X}^c$.
2. **II** has a strategy in $K[a, B]$ to reach $\mathcal{X}$.

Definition

A set $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is *Kastanas Ramsey* if for all $A \in [\mathbb{N}]^\infty$ and $a \in [\mathbb{N}]^{<\infty} \restriction A$, there exists some $B \in [a, A]$ such that one of the following holds:

1. **I** has a strategy in $K[a, B]$ to reach $\mathcal{X}^c$.
   (Strategically Ramsey: $F[a, B]$ instead of $K[a, B]$)
2. **II** has a strategy in $K[a, B]$ to reach $\mathcal{X}$.
   (Strategically Ramsey: $G[a, B]$ instead of $K[a, B]$)

Theorem (Kastanas, 1983)

A set $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is Ramsey iff it is Kastanas Ramsey.

By the Borel determinacy theorem for games with reals, we get:

Corollary

Every Borel subset of $[\mathbb{N}]^\infty$ is Ramsey.

Since analytic determinacy is not provable without large cardinals, we cannot immediately conclude that every analytic subset of $[\mathbb{N}]^\infty$ is Ramsey (even though it is true).

There is a straightforward generalisation to a **wA2**-space $(\mathcal{R}, \leq, r)$.

Definition (Cano–Di Prisco, 2023)

Let $A \in \mathcal{R}$ and $a \in \mathcal{AR}\restriction A$. The *Kastanas game* played below $[a, A]$, denoted as $K[a, A]$, is:

I	$A_0 \in [a, A]$		$A_1 \in [a_1, B_0]$	$\cdots$
II		$a_1 \in r_{\mathrm{lh}(a)+1}[a, A_0]$ $B_0 \in [a_1, A_0]$		$a_2 \in r_{\mathrm{lh}(a)+2}[a_1, A_1]$ $B_1 \in [a_2, A_1]$

The outcome of this game is $\lim_{n\to\infty} a_n$, i.e. the unique $B \in \mathcal{R}$ such that $r_{\mathrm{lh}(a)+n}(B) = a_n$ for all n.

Definition

A set $\mathcal{X} \subseteq \mathcal{R}$ is *Kastanas Ramsey* if for all $A \in \mathcal{R}$ and $a \in \mathcal{AR}\restriction A$, there exists some $B \in [a, A]$ such that one of the following holds:

1. **I** has a strategy in $K[a, B]$ to reach $\mathcal{X}^c$.
2. **II** has a strategy in $K[a, B]$ to reach $\mathcal{X}$.

Cano and Di Prisco showed that for a class of topological Ramsey spaces, a set is Ramsey iff it is Kastanas Ramsey.

Theorem (Y., 2024)

If $(\mathcal{R}, \leq, r)$ is a topological Ramsey space, then $\mathcal{X} \subseteq \mathcal{R}$ is Ramsey iff it is Kastanas Ramsey.

Again, by the Borel determinacy theorem for games with reals:

Corollary

Let $(\mathcal{R}, \leq, r)$ be a topological Ramsey space such that $\mathcal{AR}$ is countable. Then every Borel subset of $\mathcal{R}$ is Ramsey.

So for $([\mathbb{N}]^\infty, \subseteq, r)$ and $(E^{[\infty]}, \leq, r)$ with $|\mathbb{F}| = 2$, Kastanas Ramsey sets are precisely the Ramsey sets. What about $(E^{[\infty]}, \leq, r)$ with $|\mathbb{F}| > 2$ (which are also **wA2**-spaces)?

Theorem (Y., 2024)

A subset $\mathcal{X} \subseteq E^{[\infty]}$ is Kastanas Ramsey iff it is strategically Ramsey.

Let's recall the two results about analytic sets.

Theorem (Todorčević, 2010)

Let $(\mathcal{R}, \leq, r)$ be a topological Ramsey space such that $\mathcal{AR}$ is countable. Then every analytic set is Ramsey.

Theorem (Rosendal, 2010)

Every analytic subset of $E^{[\infty]}$ is strategically Ramsey.

Question. Can we unify these two statements into a single statement about analytic subsets of a **wA2**-space?

Theorem (Y., 2024)

Let $(\mathcal{R}, \leq, r)$ be a **wA2***-space such that $\mathcal{AR}$ is countable. Then every analytic subset of $\mathcal{R}$ is Kastanas Ramsey.*

We shall spend the remainder of the talk discussing the idea of the proof.

Let $(\mathcal{R}, \leq, r)$ be a **wA2**-space such that $\mathcal{AR}$ is countable. We consider the triple $(\mathcal{R} \times 2^{\mathbb{N}}, \preceq, r)$ in the following manner:

1. $(A, u) \preceq (B, v) \iff A \leq B$.
2. $r_n(A, u) = (r_n(A), u \restriction n)$.

Note that $\preceq$ is a quasi-order, but it is *not* a partial order.

Lemma

*$(\mathcal{R} \times 2^{\mathbb{N}}, \preceq, r)$ is a **wA2**-space that does not satisfy **A4**.*

This is because we can partition $\mathcal{R} \times 2^{\mathbb{N}}$ into two sets depending on the first binary bit of each $(A, u) \in \mathcal{R} \times 2^{\mathbb{N}}$. That is, we define:

$$\mathcal{X} := \{(A, u) \in \mathcal{R} \times 2^{\mathbb{N}} : u(0) = 0\}.$$

Then $\mathcal{X}$ is a clopen non-Ramsey subset of $\mathcal{R} \times 2^{\mathbb{N}}$.

Let $\pi_0 : \mathcal{R} \times 2^{\mathbb{N}} \to \mathcal{R}$ be the projection map onto the first coordinate (i.e. $\pi_0(A, u) = A$ for all $(A, u) \in \mathcal{R} \times 2^{\mathbb{N}}$).

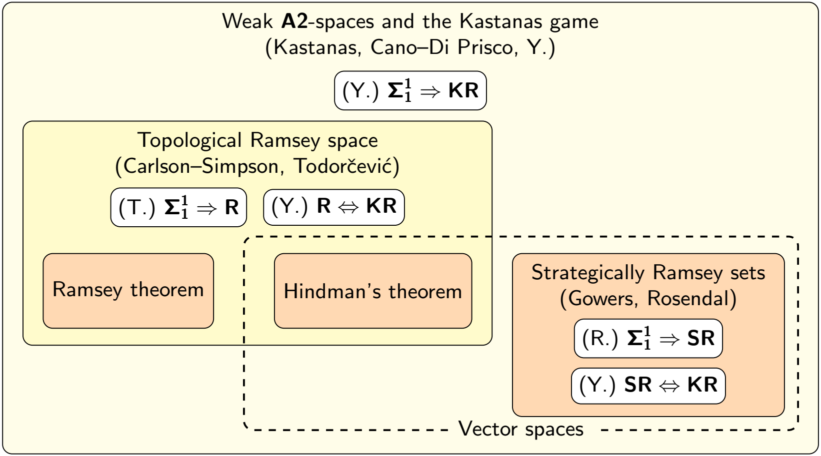
Theorem (Y., 2024)

If $\mathcal{C} \subseteq \mathcal{R} \times 2^{\mathbb{N}}$ is Kastanas Ramsey, then $\pi_0``(\mathcal{C}) \subseteq \mathcal{R}$ is also Kastanas Ramsey.

We can finish the proof in the following manner:

- Let $\mathcal{X} \subseteq \mathcal{R}$ be an analytic set.
- Let $\mathcal{C} \subseteq \mathcal{R} \times 2^{\mathbb{N}}$ be a Borel set such that $\mathcal{X} = \pi``(\mathcal{C})$.
- By the Borel determinacy theorem for games with reals, $\mathcal{C}$ is Kastanas Ramsey.
- By the above theorem, $\mathcal{X} = \pi``(\mathcal{C})$ is Kastanas Ramsey.

Thanks for listening!



R = Ramsey, **KR** = Kastanas Ramsey, **SR** = Strategically Ramsey, $\boldsymbol{\Sigma}^1_1$ = Analytic.