Weak A2 spaces, the Kastanas game and strategically Ramsey sets

Clement Yung, University of Toronto

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Ramsey Theory

In general, Ramsey theory addresses the following types of questions:

Let X be a set, and let $Y_0, \dots, Y_{n-1}$ be a partition of X. Does there exist some $i < n$ such that Y_i contains some substructure of interest?

Example.

1. X is infinite.
2. Structure = infinite set.

Fact (Pigeonhole principle)

Let X be an infinite set. If $Y_0, \ldots, Y_{n-1}$ is a partition of X, then there exists some $i < n$ such that Y_i is infinite.

Example.

1. $X = [\mathbb{N}]^n$.
2. Structure = homogeneous set, i.e. $[H]^n$ for some infinite $H \subseteq \omega$.

Theorem (Ramsey)

If $Y_0, \dots, Y_{n-1}$ is a partition of $[\mathbb{N}]^n$, then there exists some $i < n$ and an infinite $H \subseteq \omega$ such that $[H]^n \subseteq Y_i$.

Example. Let $\mathbb{F}$ be a countable (possibly finite) field. Let E be a vector space over $\mathbb{F}$ of dimension $\aleph_0$, with Hamel basis $(e_n)_{n<\omega}$. Given a vector $x \in E$, we may write

$$x = \sum_{n<\omega} \lambda_n(x) e_n,$$

where only finitely many λ_n's are non-zero. We may then write:

$$\operatorname{supp}(x) := \{n < \omega : \lambda_n(x) \neq 0\}.$$

Example

If $x = 2e_3 - 6e_{17} + 5e_{58}$, then $\operatorname{supp}(x) = \{3, 17, 58\}$.

Notation

Given two vectors x, y we write:

$$x < y \iff \max(\mathrm{supp}(x)) < \min(\mathrm{supp}(y)).$$

Example

If:

1. $x = 2e_3 - 6e_{17} + 5e_{58}$,
2. $y = 5e_{67} + 990e_{133} - 155e_{236}$,
3. $z = -32e_{43} + 5e_{665}$,

then $x < y$ but $x \not< z$.

Definition

An *infinite block sequence* is a $<$-increasing sequence of elements of E.

Definition

An infinite-dimensional subspace $V \subseteq W$ is a *block subspace* if $V = \operatorname{span}\{x_n : n < \omega\}$ for some infinite block sequence $(x_n)_{n<\omega}$. Note that $\{x_n\}_{n<\omega}$ is a (unique) basis of V.

Fact

Every infinite-dimensional subspace of E contains an infinite-dimensional block subspace.

Now consider the following setting:

1. $X = E \setminus \{0\}$, the set of non-zero vectors.
2. Structure = infinite-dimensional block subspaces (without 0).

Does the Ramsey theorem hold for this variant?

Theorem (Hindman)

Suppose that $|\mathbb{F}| = 2$. If $Y_0, \ldots, Y_{n-1}$ is a partition of $E \setminus \{0\}$, then there exists some $i < n$ and some infinite-dimensional block subspace V such that $V \setminus \{0\} \subseteq Y_i$.

This theorem fails if $|\mathbb{F}| > 2$. We define the set Y as:

$$\begin{aligned} &\{x \in E \setminus \{0\} : x = e_n + y \text{ for some } e_n < y\}, \\ = &\{x \in E \setminus \{0\} : x = e_{n_0} + \lambda n_1 e_{n_1} + \cdots + \lambda_{n_k} e_{n_k} \text{ and } n_0 < \cdots < n_k\}. \end{aligned}$$

Then Y, Y^c partitions $E \setminus \{0\}$, but neither Y nor Y^c contains an infinite-dimensional subspace.

Example

Let $A = (e_0 + e_1, e_2 + e_3, \dots)$. Then $e_0 + e_1 \in Y$, but $2e_0 + 2e_1 \in Y^c$, so $\operatorname{span}(A) \setminus \{0\}$ is not a subset of either Y or Y^c.

Infinite-dimensional Ramsey theory

Infinite-dimensional Ramsey theory addresses a similar type of question, but instead, we partition $X^{\mathbb{N}}$ or a closed subest $\mathcal{R}$ of $X^{\mathbb{N}}$. Here we equip X with the discrete topology, and $X^{\mathbb{N}}$ with the product topology.

More precisely:

Let X be a set. Let $\mathcal{X}_0, \dots, \mathcal{X}_{n-1}$ be a partition of $\mathcal{R}$, a closed subset of $X^{\mathbb{N}}$. Does there exist some $i < n$ such that $\mathcal{X}_i$ contains some substructure of interest?

Example.

1. $X = \mathbb{N}$.
2. $\mathcal{R} = [\mathbb{N}]^\infty$, which may be identified as the set of strictly increasing sequences in $\mathbb{N}^\mathbb{N}$.
3. Structure = Ellentuck neighbourhood of infinite subset.

Let $[\mathbb{N}]^{<\infty}\restriction A := \{a \in [\mathbb{N}]^{<\infty} : a \subseteq A\}$.

Notation

Given $A \in [\mathbb{N}]^{\infty}$ and $a \in [\mathbb{N}]^{<\infty}\restriction A$, we write:

$$[a, A] := \{B \in [\mathbb{N}]^{\infty} : a \sqsubseteq B \text{ and } B \subseteq A\}$$

where $a \sqsubseteq B$ means that $B \cap \max(a) = a$.

Each $[a, A]$ is also called an *Ellentuck neighbourhood*.

Definition

A set $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is *Ramsey* if for all $A \in [\mathbb{N}]^\infty$ and $a \in [\mathbb{N}]^{<\infty} \restriction A$, there exists some $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \subseteq \mathcal{X}^c$.

Theorem (Galvin-Prikry)

If $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is Borel, then $\mathcal{X}$ is Ramsey.

Example. Let E a vector space over a countable field $\mathbb{F}$ of dimension $\aleph_0$, with Hamel basis $(e_n)_{n<\omega}$.

1. $X = E$.
2. $\mathcal{R} = E^{[\infty]}$, the set of all infinite block sequences of vectors ($\Leftrightarrow$ infinite-dimensional block subspaces).
3. Structure = Ellentuck neighbourhood of infinite-dimensional block subspaces.

Notation

If $A = (x_n)_{n<\omega}$ and $B = (y_n)_{n<\omega}$ are two elements of $E^{[\infty]}$, then we write:

$$B \leq A \iff \text{span}(B) \subseteq \text{span}(A).$$

Notation

Given $A = (x_n)_{n<\omega} \in E^{[\infty]}$, let $E^{[<\infty]}\restriction A$ be the set of finite block subspaces of A. In other words, the set of $(y_m)_{m<N}$ such that $\operatorname{span}\{y_0, \ldots, y_{N-1}\} \subseteq \operatorname{span}(A)$.

Notation

Given $A \in E^{[\infty]}$ and $a \in E^{[<\infty]}\restriction A$, we write:

$$[a, A] := \{B \in E^{[\infty]} : a \sqsubseteq B \text{ and } B \leq A\}$$

where $a \sqsubseteq B$ means that a is an initial segment of B.

Definition

A set $\mathcal{X} \subseteq E^{[\infty]}$ is *Ramsey* if for all $A \in E^{[\infty]}$ and $a \in E^{[<\infty]} \upharpoonright A$, there exists some $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \subseteq \mathcal{X}^c$.

Theorem (Infinite-dimensional Hindman's theorem)

Suppose that $|\mathbb{F}| = 2$. If $\mathcal{X} \subseteq E^{[\infty]}$ is Borel, then it is Ramsey.

Again, this theorem fails for $|\mathbb{F}| > 2$ - there exists a clopen subset $\mathcal{X}$ of $E^{[\infty]}$ which is not Ramsey.

We observe some patterns:

1. $\mathcal{R}$ is a set of infinite increasing sequences under some partial order $<$.
2. Using either $\subseteq$ or $\leq$, we defined the Ellentuck neighbourhood $[a, A]$, and the notion of Ramsey subsets $\mathcal{X} \subseteq \mathcal{R}$.

Question. Can this pattern be captured and made into an abstract framework?

Topological Ramsey theory

Consider the following setting:

1. $\mathcal{R}$ is a non-empty set, representing some set of infinite increasing sequences.
2. $\leq$ is a quasi-order on $\mathcal{R}$.
3. $\mathcal{AR}$ is a non-empty set, representing the set of finite increasing sequences.
    - For $\mathcal{R} = [\mathbb{N}]^\infty$, $\mathcal{AR} = [\mathbb{N}]^{<\infty}$.
    - For $\mathcal{R} = E^{[\infty]}$, $\mathcal{AR} = E^{[<\infty]}$.
4. $r : \mathcal{R} \times \omega \to \mathcal{AR}$ is a function, with $r_n(-) := r(-, n)$, is a restriction map that takes the first n elements of the sequence.
    - If $A = \{x_0, x_1, \dots\} \in [\mathbb{N}]^\infty$, then $r_n(A) = \{x_0, \dots, x_{n-1}\}$.
    - If $A = (x_0, x_1, \dots) \in E^{[\infty]}$, then $r_n(A) = (x_0, \dots, x_{n-1})$.

Notation

Given a triple $(\mathcal{R}, \leq, r)$, for $A \in \mathcal{R}$ and $a \in \mathcal{AR}$:

$$a \sqsubseteq A \iff a = r_n(A) \text{ for some } n \in \mathbb{N}.$$

Notation

If $A \in \mathcal{R}$ and $a \in \mathcal{AR}$,

$$[a, A] := \{B \in \mathcal{AR} : a \sqsubseteq B \text{ and } B \leq A\}.$$

Axiom (**A1**, Sequencing)

(1) $r_0(A) = \emptyset$ for all $A \in \mathcal{AR}$.

(2) $A \neq B$ implies $r_n(A) \neq r_n(B)$ for some n.

(3) $r_n(A) = r_m(B)$ implies $n = m$ and $r_k(A) = r_k(B)$ for all $k < n$.

Axiom (**A2**, Finitisation)

There is a quasi-ordering $\leq_{\text{fin}}$ on $\mathcal{AR}$ such that:

(1) $\{b \in \mathcal{AR} : b \leq_{\text{fin}} a\}$ is finite for all $b \in \mathcal{AR}$.

(2) $A \leq B$ iff $\forall n \exists m [r_n(A) \leq_{\text{fin}} r_m(B)]$.

(3) $\forall a, b \in \mathcal{AR}[a \sqsubseteq b \wedge b \leq_{\text{fin}} c \rightarrow \exists d \sqsubseteq c[a \leq_{\text{fin}} d]]$.

1. $([\mathbb{N}]^\infty, \subseteq, r)$ satisfies **A2**, as for all $a \in [\mathbb{N}]^{<\infty}$, $\{b : b \subseteq a\}$ is finite.
2. $(E^{[\infty]}, \leq, r)$ satisfies **A2** if $|\mathbb{F}| < \infty$, as if $a = (x_i)_{i<n} \in E^{[<\infty]}$, then there are finitely many subspaces of a.
3. $(E^{[\infty]}, \leq, r)$ does not satisfy **A2** if $|\mathbb{F}| = \infty$, as if $a = (x_0, x_1)$, then $\text{span}\{\lambda x_0 + x_1\}$ is a (block) subspace for any $\lambda \in \mathbb{F}$.

Axiom (**A3**, Amalgamation)

The depth function defined by, for $B \in \mathcal{R}$ and $a \in \mathcal{AR}$:

$$\operatorname{depth}_B(a) := \begin{cases} \min\{n < \omega : a \leq_{\text{fin}} r_n(B)\}, & \text{if such } n \text{ exists} \\ \infty, & \text{otherwise} \end{cases}$$

satisfies the following:

(1) If $\operatorname{depth}_B(a) < \infty$, then for all $A \in [\operatorname{depth}_B(a), B]$, $[a, A] \neq \emptyset$.

(2) $A \leq B$ and $[a, A] \neq \emptyset$ imply that there exists $A' \in [\operatorname{depth}_B(a), B]$ such that $\emptyset \neq [a, A'] \subseteq [a, A]$.

We let $\mathcal{AR}_n$ be the image of the map $r_n(-)$, i.e. the set of all finite approximations of length n.

Axiom (A4, Pigeonhole)

If $\text{depth}_B(a) < \infty$ and if $\mathcal{O} \subseteq \mathcal{AR}_{\text{lh}(a)+1}$, then there exists $A \in [\text{depth}_B(a), B]$ such that $r_{\text{lh}(a)+1}[a, A] \subseteq \mathcal{O}$ or $r_{\text{lh}(a)+1}[a, A] \subseteq \mathcal{O}^c$.

1. $([\mathbb{N}]^\infty, \subseteq, r)$ satisfies **A4**, due to the pigeonhole principle.
2. $(E^{[\infty]}, \leq, r)$ satisfies **A4** if $|\mathbb{F}| = 2$, due to Hindman's theorem.
3. $(E^{[\infty]}, \leq, r)$ does not satisfy **A4** if $|\mathbb{F}| > 2$, as Hindman's theorem fails for $|\mathbb{F}| > 2$.

Observe that every $A \in \mathcal{R}$ can be uniquely identified as the sequence $(r_n(A))_{n<\omega}$, an element of $\mathcal{AR}^{\mathbb{N}}$. Therefore, we may identify $\mathcal{R}$ as a subset of $\mathcal{AR}^{\mathbb{N}}$.

Recall that in infinite-dimensional Ramsey theory, we require $\mathcal{R}$ to be a closed subset of $X^{\mathbb{N}}$. We make a similar requirement here.

Definition

$(\mathcal{R}, \leq, r)$ is a *closed triple* if for all $\sqsubseteq$-increasing sequence $(a_n)_{n<\omega}$ of elements in $\mathcal{AR}$ such that $\mathrm{lh}(a_n) = n$, there exists some $A \in \mathcal{R}$ such that $r_n(A) = a_n$ for all n.

In other words, $\mathcal{R}$ is a metrically closed subset of $\mathcal{AR}^{\mathbb{N}}$.

Definition

A *topological Ramsey space* is a closed triple $(\mathcal{R}, \leq, r)$ satisfying **A1**-**A4**.

Definition

Let $(\mathcal{R}, \leq, r)$ be a topological Ramsey space. A set $\mathcal{X} \subseteq \mathcal{R}$ is *Ramsey* if for all $A \in \mathcal{R}$ and $a \in \mathcal{AR}{\restriction}A$, there exists some $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \subseteq \mathcal{X}^c$.

Theorem (Todorčević)

Let $(\mathcal{R}, \leq, r)$ be a topological Ramsey space. If $\mathcal{AR}$ is countable and $\mathcal{X} \subseteq \mathcal{R}$ is Borel, then $\mathcal{X}$ is Ramsey.

Weak **A2** spaces

Although countable vector spaces fail to satisfy the axioms **A1**-**A4** for $|\mathbb{F}| > 2$, a rich Ramsey theory of countable vector spaces has been developed in the past 20 years with lots of similarities to topological Ramsey theory.

Question. Is there an overarching framework that encompasses topological Ramsey theory and the Ramsey theory of countable vector spaces?

Axiom (wA2, Weak Finitisation)

There is a quasi-ordering $\leq_{\mathrm{fin}}$ on $\mathcal{AR}$ such that:

(w1) $\{b \in \mathcal{AR} : b \leq_{\mathrm{fin}} a\}$ is countable for all $b \in \mathcal{AR}$.

(2) $A \leq B$ iff $\forall n \exists m [r_n(A) \leq_{\mathrm{fin}} r_m(B)]$.

(3) $\forall a, b \in \mathcal{AR} [a \sqsubseteq b \wedge b \leq_{\mathrm{fin}} c \rightarrow \exists d \sqsubseteq c [a \leq_{\mathrm{fin}} d]]$.

Definition

A triple $(\mathcal{R}, \leq, r)$ is a *weak* **A2** *space*, or just **wA2**-*space*, if it is a closed triple satisfying **A1**, **wA2**, **A3**.

Thus, topological Ramsey spaces and countable vector spaces are examples of **wA2**-spaces.

Abstract Kastanas Game

We discuss one application of **wA2**-spaces by introducing the abstract Kastanas game. Unless stated otherwise, we assume that $(\mathcal{R}, \leq, r)$ is a **wA2**-space.

Definition (Kastanas, Cano-Di Prisco)

Let $A \in \mathcal{R}$ and $a \in \mathcal{AR}{\restriction}A$. The *Kastanas game* played below $[a, A]$, denoted as $K[a, A]$, is:

I	$A_0 \in [a, A]$		$A_1 \in [a_1, B_0]$	$\cdots$
II		$a_1 \in r_{\mathrm{lh}(a)+1}[a, A_0]$ $B_0 \in r_{\mathrm{lh}(a_1)+1}[a_1, A_0]$		$a_2 \in [a_1, A_1]$ $B_1 \in [a_2, A_1]$

The outcome of this game is $\lim_{n\to\infty} a_n$, i.e. the unique $B \in \mathcal{R}$ such that $r_{\mathrm{lh}(a)+n}(B) = a_n$ for all n.

Definition

We say that **I** (similarly **II**) has a strategy in $K[a, A]$ to *reach* $\mathcal{X} \subseteq \mathcal{R}$ if it has a strategy in $K[a, A]$ to ensure the outcome is in $\mathcal{X}$.

Definition

A set $\mathcal{X} \subseteq \mathcal{R}$ is *Kastanas Ramsey* if for all $A \in \mathcal{R}$ and $a \in \mathcal{AR}{\restriction}A$, there exists some $B \in [a, A]$ such that one of the following holds:

1. **I** has a strategy in $K[a, B]$ to reach $\mathcal{X}^c$.
2. **II** has a strategy in $K[a, B]$ to reach $\mathcal{X}$.

Definition

We say that **I** (similarly **II**) has a strategy in $K[a, A]$ to *reach* $\mathcal{X} \subseteq \mathcal{R}$ if it has a strategy in $K[a, A]$ to ensure the outcome is in $\mathcal{X}$.

Definition

A set $\mathcal{X} \subseteq \mathcal{R}$ is *Kastanas Ramsey* if for all $A \in \mathcal{R}$ and $a \in \mathcal{AR}{\upharpoonright}A$, there exists some $B \in [a, A]$ such that one of the following holds:

1. **I** has a strategy in $K[a, B]$ to reach $\mathcal{X}^c$.
   (Definition of Ramsey: $[a, B] \subseteq \mathcal{X}^c$.)
2. **II** has a strategy in $K[a, B]$ to reach $\mathcal{X}$.
   (Definition of Ramsey: $[a, B] \subseteq \mathcal{X}$.)

Theorem (Kastanas)

A set $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is Ramsey iff Kastanas Ramsey.

1. By the Borel determinacy for Polish spaces, we have that every Borel subset of $[\mathbb{N}]^\infty$ is Kastanas Ramsey.
2. By Kastanas' theorem, we can conclude the Galvin-Prikry theorem, i.e. every Borel subset of $[\mathbb{N}]^\infty$ is Ramsey.

Question. Can we generalise this fact to topological Ramsey spaces?

Theorem (Y.)

If $(\mathcal{R}, \leq, r)$ is a closed triple satisfying **A1**-**A4**, *then $\mathcal{X} \subseteq \mathcal{R}$ is Ramsey iff it is Kastanas Ramsey.*

1. By the Borel determinacy of Polish spaces, we can conclude that if $\mathcal{AR}$ is countable, then every Borel subset of $\mathcal{R}$ is Kastanas Ramsey.
2. Since Kastanas Ramsey $\iff$ Ramsey, we get Todorčević's theorem that every Borel subset of $\mathcal{R}$ is Ramsey.

What about analytic sets?

Theorem (Mathias-Silver)

Every analytic subset of $[\mathbb{N}]^\infty$ is Ramsey.

Theorem (Todorčević)

Let $(\mathcal{R}, \leq, r)$ be a topological Ramsey space, and assume that $\mathcal{AR}$ is countable. Then every analytic subset of $\mathcal{R}$ is Ramsey.

Since analytic determinacy is not a theorem of ZFC, it's not clear that the equivalence between Kastanas Ramsey sets and Ramsey sets implies both theorems.

Good news. We can use the equivalence to prove both theorems.

To simplify things, we shall demonstrate this for $[\mathbb{N}]^\infty$.

Goal. Provide a proof of the Mathias-Silver theorem in the following steps:

1. Define a version of the Kastanas game (and Kastanas Ramsey sets) on $[\mathbb{N}]^\infty \times 2^\infty$. By the Borel determinacy for Polish spaces, all Borel subsets of $[\mathbb{N}]^\infty \times 2^\infty$ are Kastanas Ramsey.
2. Show that Kastanas Ramsey sets are closed under projections. Therefore, analytic subsets of $[\mathbb{N}]^\infty$ are Kastanas Ramsey.
3. By Kastanas' theorem, analytic subsets of $[\mathbb{N}]^\infty$ are Ramsey.

Kastanas game on $[\mathbb{N}]^\infty \times 2^\infty$

Definition

Let $A \in [\mathbb{N}]^\infty$, and let $a \in [\mathbb{N}]^{<\infty}$ and $p \in 2^{|a|}$. The *Kastanas game* played below $[a, A, p]$, denoted as $K[a, A, p]$, is:

I	$A_0 = A$	$A_1 \subseteq B_0$	$\cdots$
II	$x_0 \in A_0$	$x_1 \in A_1$	$\cdots$
	$\varepsilon_0 \in \{0, 1\}$	$\varepsilon_1 \in \{0, 1\}$	$\cdots$
	$B_0 \subseteq A_0$	$B_1 \subseteq A_1$	$\cdots$

where:

- $\max(a) < x_0 < x_1 < \cdots$.
- A_n, B_n are infinite subsets of $\mathbb{N}$.

The outcome of the game is
$(a \cup \{x_0, x_1, \ldots\}, p^\frown(\varepsilon_0, \varepsilon_1, \ldots)) \in [a, A] \times 2^\infty$.

Definition

We say that **I** (similarly **II**) has a strategy in $K[a, A, p]$ to *reach* $\mathcal{C} \subseteq [\mathbb{N}]^\infty \times 2^\infty$ if it has a strategy in $K[a, A, p]$ to ensure the outcome is in $\mathcal{C}$.

Definition

A set $\mathcal{C} \subseteq [\mathbb{N}]^\infty \times 2^\infty$ is *Kastanas Ramsey* if for all $A \in [\mathbb{N}]^\infty$, $a \in [\mathbb{N}]^{<\infty} \upharpoonright A$ and $p \in 2^{|a|}$, there exists some $B \in [a, A]$ such that one of the following holds:

1. **I** has a strategy in $K[a, B, p]$ to reach $\mathcal{C}^c$.
2. **II** has a strategy in $K[a, B, p]$ to reach $\mathcal{C}$.

Let $\pi_0 : [\mathbb{N}]^\infty \times 2^\infty \to [\mathbb{N}]^\infty$ be the projection to the first coordinate.

Theorem

If $\mathcal{C} \subseteq [\mathbb{N}]^\infty \times 2^\infty$ is Kastanas Ramsey, then $\pi_0[\mathcal{C}] \subseteq [\mathbb{N}]^\infty$ is Kastanas Ramsey.

We split the proof of the theorem into two lemmas.

Lemma

Let $\mathcal{C} \subseteq [\mathbb{N}]^\infty \times 2^\infty$ be a subset. Let $A \in [\mathbb{N}]^\infty$, $a \in [\mathbb{N}]^{<\infty}\restriction A$. If **II** *has a strategy in $K[a, A, p]$ to reach $\mathcal{C}$ for some $p \in 2^{\mathrm{lh}(a)}$, then* **II** *has a strategy in $K[a, A]$ to reach $\pi_0[\mathcal{C}]$.*

Proof.

The strategy by **II** in the game $K[a, A, p]$ to reach $\mathcal{C}$, with the ε_n's ignored, is a strategy for **II** in $K[a, A]$ to reach $\pi_0[\mathcal{C}]$. □

Lemma

*Let $\mathcal{C} \subseteq [\mathbb{N}]^\infty \times 2^\infty$ be a subset. Let $A \in [\mathbb{N}]^\infty$, $a \in [\mathbb{N}]^{<\infty} \restriction A$. If for all $p \in 2^{\mathrm{lh}(a)}$, there exists some $C \in [a, A]$ such that **I** has a strategy in $K[a, C, p]$ to reach $\mathcal{C}^c$, then there exists some $B \in [a, A]$ such that **I** has a strategy in $K[a, B]$ to reach $\pi_0[\mathcal{C}]^c$.*

Since $\pi_0[\mathcal{C}^c] \neq \pi_0[\mathcal{C}]^c$ in general, the same naive argument doesn't work here.

In the interest of time, we shall prove this lemma only for $a = \emptyset$.

Proof of the second Lemma

Let $B \in [A]^\infty$ and σ be a strategy for **I** in $K[\emptyset, B, \emptyset]$ (in $[\mathbb{N}]^\infty \times 2^\infty$) to reach $\mathcal{C}^c$. How do we define a strategy τ for **I** in $K[\emptyset, B]$ (in $[\mathbb{N}]^\infty$) to reach $\pi_0[\mathcal{C}]^c$?

- Say that the outcome of a complete run in $K[\emptyset, B]$ (in $[\mathbb{N}]^\infty$), following τ, is $D = \{x_0, x_1, \dots\}$.
- $D \in \pi_0[\mathcal{C}]^c$ iff for all $x \in 2^\infty$, $(D, x) \in \mathcal{C}^c$.
- **Goal.** Design τ such that, for any outcome D and any $x \in 2^\infty$ (in $[\mathbb{N}]^\infty$), there is a simulation of the game in $K[\emptyset, B, \emptyset]$ (in $[\mathbb{N}]^\infty \times 2^\infty$) following σ, such that the outcome is (D, x). By our choice of σ, $(D, x) \in \mathcal{C}^c$.

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$
II	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$
II	

or

I	$A_0 = B$
II	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$

or

I	$A_0 = B$
II	$x_0 \in A_0$ $\varepsilon_0 = 1$

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$
II	$x_0 \in A_0$
	$B_0 \subseteq A_0$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$
II	$x_0 \in A_0$
	$\varepsilon_0 = 0$
	$B_0 \subseteq A_0$

or

I	$A_0 = B$
II	$x_0 \in A_0$
	$\varepsilon_0 = 1$

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	

or

I	$A_0 = B$
II	$x_0 \in A_0$ $\varepsilon_0 = 1$

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	

or

I	$A_0 = B$
II	$x_0 \in A_0$ $\varepsilon_0 = 1$ $A_1^0 \subseteq A_0$

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	
II		$x_0 \in A_0$
		$B_0 \subseteq A_0$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$		$A_1^0 := \sigma(x_0, 0, B_0)$
II		$x_0 \in A_0$	
		$\varepsilon_0 = 0$	
		$B_0 \subseteq A_0$	

or

I	$A_0 = B$		$A_1^1 := \sigma(x_0, 1, A_1^0)$
II		$x_0 \in A_0$	
		$\varepsilon_0 = 1$	
		$A_1^0 \subseteq A_0$	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$ $\qquad\qquad$ $\tau(x_0, B_0) := A_1^1$
II	$x_0 \in A_0$
	$B_0 \subseteq A_0$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$ $\qquad\qquad$ $A_1^0 := \sigma(x_0, 0, B_0)$
II	$x_0 \in A_0$
	$\varepsilon_0 = 0$
	$B_0 \subseteq A_0$

or

I	$A_0 = B$ $\qquad\qquad$ $A_1^1 := \sigma(x_0, 1, A_1^0)$
II	$x_0 \in A_0$
	$\varepsilon_0 = 1$
	$A_1^0 \subseteq A_0$

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$	$x_1 \in A_1$ $B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ:

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	

or

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$
II	$x_0 \in A_0$ $\varepsilon_0 = 1$ $A_1^0 \subseteq A_0$	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$
II	$x_0 \in A_0$	$x_1 \in A_1$
	$B_0 \subseteq A_0$	$B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 0)$:

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$
II	$x_0 \in A_0$	$x_1 \in A_1^1$
	$\varepsilon_0 = 0$	$\varepsilon_1 = 0$
	$B_0 \subseteq A_0$	

or

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$
II	$x_0 \in A_0$	$x_1 \in A_1^1$
	$\varepsilon_0 = 0$	$\varepsilon_1 = 1$
	$B_0 \subseteq A_0$	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$	
II	$x_0 \in A_0$ $B_0 \subseteq A_0$		$x_1 \in A_1$ $B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 0)$:

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$		$A_2^0 := \sigma(x_0, 0, B_0, x_1, 0, B_1)$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$		$x_1 \in A_1^1$ $\varepsilon_1 = 0$ $B_1 \subseteq A_1^1$	

or

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$	
II	$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$		$x_1 \in A_1^1$ $\varepsilon_1 = 1$

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$	$x_1 \in A_1$ $B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 0)$:

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$	$A_2^0 := \sigma(x_0, 0, B_0, x_1, 0, B_1)$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	$x_1 \in A_1^1$ $\varepsilon_1 = 0$ $B_1 \subseteq A_1^1$	

or

I	$A_0 = B$	$A_1^0 := \sigma(x_0, 0, B_0)$	$A_2^1 := \sigma(x_0, 0, B_0, x_1, 1, A_2^0)$
II	$x_0 \in A_0$ $\varepsilon_0 = 0$ $B_0 \subseteq A_0$	$x_1 \in A_1^1$ $\varepsilon_1 = 1$ $A_2^0 \subseteq A_1^1$	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$
II	$x_0 \in A_0$	$x_1 \in A_1$
	$B_0 \subseteq A_0$	$B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 1)$:

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$
II	$x_0 \in A_0$	$x_1 \in A_1^1$
	$\varepsilon_0 = 1$	$\varepsilon_1 = 0$
	$A_1^0 \subseteq A_0$	$A_2^1 \subseteq A_1^1$

or

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$
II	$x_0 \in A_0$	$x_1 \in A_1^1$
	$\varepsilon_0 = 1$	$\varepsilon_1 = 1$
	$A_1^0 \subseteq A_0$	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$		$\tau(x_0, B_0) := A^1_1$	
II		$x_0 \in A_0$ $B_0 \subseteq A_0$		$x_1 \in A_1$ $B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 1)$:

I	$A_0 = B$		$A^1_1 := \sigma(x_0, 1, A^0_1)$		$A^2_2 := \sigma(x_0, 1, B_0, x_1, 0, A^1_2)$
II		$x_0 \in A_0$ $\varepsilon_0 = 1$ $A^0_1 \subseteq A_0$		$x_1 \in A^1_1$ $\varepsilon_1 = 0$ $A^1_2 \subseteq A^1_1$	

or

I	$A_0 = B$		$A^1_1 := \sigma(x_0, 1, A^0_1)$	
II		$x_0 \in A_0$ $\varepsilon_0 = 1$ $A^0_1 \subseteq A_0$		$x_1 \in A^1_1$ $\varepsilon_1 = 1$

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$
II	$x_0 \in A_0$ $B_0 \subseteq A_0$	$x_1 \in A_1$ $B_1 \subseteq A_1$

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 1)$:

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$	$A_2^2 := \sigma(x_0, 1, B_0, x_1, 0, A_2^1)$
II	$x_0 \in A_0$ $\varepsilon_0 = 1$ $A_1^0 \subseteq A_0$	$x_1 \in A_1^1$ $\varepsilon_1 = 0$ $A_2^1 \subseteq A_1^1$	

or

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$	$A_2^3 := \sigma(x_0, 1, A_1^0, x_1, 1, A_2^2)$
II	$x_0 \in A_0$ $\varepsilon_0 = 1$ $A_1^0 \subseteq A_0$	$x_1 \in A_1^1$ $\varepsilon_1 = 1$ $A_2^2 \subseteq A_1^1$	

$K[\emptyset, B]$, defining τ for **I**:

I	$A_0 = B$	$\tau(x_0, B_0) := A_1^1$		$\tau(x_0, B_0, x_1, B_1) := A_2^3$
II	$x_0 \in A_0$		$x_1 \in A_1$	
	$B_0 \subseteq A_0$		$B_1 \subseteq A_1$	

(Simulation) $K[\emptyset, B, \emptyset]$, **I** following σ $(\varepsilon_0 = 1)$:

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$		$A_2^2 := \sigma(x_0, 1, B_0, x_1, 0, A_2^1)$
II	$x_0 \in A_0$		$x_1 \in A_1^1$	
	$\varepsilon_0 = 1$		$\varepsilon_1 = 0$	
	$A_1^0 \subseteq A_0$		$A_2^1 \subseteq A_1^1$	

or

I	$A_0 = B$	$A_1^1 := \sigma(x_0, 1, A_1^0)$		$A_2^3 := \sigma(x_0, 1, A_1^0, x_1, 1, A_2^2)$
II	$x_0 \in A_0$		$x_1 \in A_1^1$	
	$\varepsilon_0 = 1$		$\varepsilon_1 = 1$	
	$A_1^0 \subseteq A_0$		$A_2^2 \subseteq A_1^1$	

Generalising to **wA2**-spaces

Let $(\mathcal{R}, \leq, r)$ be a **wA2**-space. In a way similar to how we go from $[\mathbb{N}]^\infty$ to $[\mathbb{N}]^\infty \times 2^\infty$, we may consider going from $\mathcal{R}$ to $\mathcal{R} \times 2^\infty$.

More precisely, we shall construct the triple $(\mathcal{R} \times 2^\infty, \preceq, r)$ in the following manner:

1. $(A, u) \preceq (B, v) \iff A \leq B$.
2. $r_n(A, u) = (r_n(A), u{\restriction}n)$.

Note that $\preceq$ is not a partial order.

Lemma

Let $(\mathcal{R}, \leq, r)$ be a **wA2***-space. Then the closed triple $(\mathcal{R} \times 2^\infty, \preceq, r)$ defined above is a* **wA2***-space which does not satisfy* **A4***.*

This means that $([\mathbb{N}]^\infty \times 2^\infty, \preceq, r)$ is a **wA2**-space, so we may consider the abstract Kastanas game on $([\mathbb{N}]^\infty \times 2^\infty, \preceq, r)$.

Fact

The abstract Kastanas game on $([\mathbb{N}]^\infty \times 2^\infty, \preceq, r)$ is precisely the "modified" Kastanas game that we presented earlier.

Theorem (Y.)

Let $(\mathcal{R}, \leq, r)$ be a **wA2***-space. If $\mathcal{C} \subseteq \mathcal{R} \times 2^{\infty}$ is Kastanas Ramsey, then $\pi_0[\mathcal{C}] \subseteq \mathcal{R}$ is Kastanas Ramsey.*

Corollary (Y.)

Let $(\mathcal{R}, \leq, r)$ be a **wA2***-space, and assume that $\mathcal{AR}$ is countable. Then every analytic subset of $\mathcal{R}$ is Kastanas Ramsey.*

Strategically Ramsey sets

Todorčević's theorem asserts that if $(\mathcal{R}, \leq, r)$ is a closed triple satisfying **A1**-**A4**, and $\mathcal{AR}$ is countable, then every analytic subset of $\mathcal{R}$ is Kastanas Ramsey. What about countable vector spaces?

Theorem (Rosendal)

Every analytic subset of $E^{[\infty]}$ is strategically Ramsey.

Proposition

A subset $\mathcal{X} \subseteq E^{[\infty]}$ is Kastanas Ramsey iff it is strategically Ramsey.

Thanks for listening!

1. The Ramsey theorem for $([\mathbb{N}]^\infty, \subseteq, r)$ (pigeonhole principle) and $(E^{[\infty]}, \leq, r)$ when $|\mathbb{F}| = 2$ are both true.
2. Todorčević developed topological Ramsey theory to provide a general framework to prove these results.
3. $(E^{[\infty]}, \leq, r)$ for $|\mathbb{F}| > 2$ is not a topological Ramsey space, but still contains a rich Ramsey theory. **wA2**-space proposes an extension of topological Ramsey theory to such spaces.
4. We defined the abstract Kastanas game for **wA2**-spaces and Kastanas Ramsey sets. For topological Ramsey spaces, Kastanas Ramsey sets are precisely Ramsey sets.
5. By considering $(\mathcal{R} \times 2^\infty, \preceq, r)$, we showed that every analytic subset of $\mathcal{R}$ is Kastanas Ramsey. This implies that every analytic subset of $E^{[\infty]}$ is strategically Ramsey.