MAT309 Introduction to Mathematical Logic - Fall 2025
Tutorial 4

This document was prepared in a hurry, and may contain errors. You can consult me in class if you have any questions.

Activity 1.

Review Problem 5.4. Decide which expression is a legitimate formula, and which is not, and why.

Solution

The **first test** is to check if a term/formula is legitimate is to see if it contains symbols that shouldn't be there. For instance, a formula should not contain any (a) quantifiers and formula "stuff", i.e. $\forall$, $\exists$, $\neg$, (and), (b) relations, e.g. $P_1^k, P_2^k, \ldots$ for Example 5.2(6), and $<, |$ for Example 5.1. If a term passes the check above, you may begin the **second test** by "dissecting" the function symbols. The general rule is to *dissect them from right to left.*

Remember that the "formula symbols", i.e. $\forall$, $\neg$ and $\to$, are "appropriately placed", so they do not need to be dissected. This does not include $=$, since the formal way to write "$t_1 = t_2$" is $= t_1 t_2$.

(1) $\underline{= 0 + v_7 \cdot 1 v_3}$: This is a legitimate formula defined by the language of Example 5.1, $\mathcal{L}_{NT}$, or Example 5.2(1), $\mathcal{L}_F$.

$$\begin{aligned} = 0 + v_7 \cdot 1 v_3 &\to= 0 + v_7 (1 \cdot v_3) && \text{Dissecting the } \cdot \\ &\to= 0 (v_7 + (1 \cdot v_3)) && \text{Dissecting the } + \\ &\to 0 = (v_7 + (1 \cdot v_3)) && \text{Dissecting the } = \end{aligned}$$

(2) $\underline{(\neg = v_1 v_1)}$: This is a legitimate formula that can be defined by any language.

$$(\neg = v_1 v_1) \to (\neg (v_1 = v_1)) \qquad \text{Dissecting the } =$$

(3) $\underline{(| v_2 0 \to \cdot 01)}$: This is not a legitimate formula defined by the language of

Example 5.1, $\mathcal{L}_{NT}$.

$$\begin{aligned}(\mid v_2 0 \to \cdot 01) &\to (|v_2 0 \to (0 \cdot 1)) && \text{Dissecting the } \cdot \\ &\to (\mid v_2 0 \to (0 \cdot 1)) && \text{Dissecting the } \cdot \\ &\to (v_2 \mid 0 \to (0 \cdot 1)) && \text{Dissecting the } \mid\end{aligned}$$

This intuitively doesn't make sense: $0 \cdot 1$ is a term, not a formula, but for a formula of the form $(\alpha \to \beta)$ to be valid, both α and β have to be formulas.

(4) $\underline{(\neg \forall v_5 (= v_5 v_5))}$: This is a legitimate formula that can be defined by any language. There may be a lot of symbols, but observe that $=$ is the only one that needs dissecting.

$$(\neg \forall v_5 (= v_5 v_5)) \to (\neg \forall v_5 ((v_5 = v_5)))\text{Dissecting the } =$$

The double brackets are not a typo. In fact, this formula is not legitimate precisely because of the unnecessary brackets! While $= v_5 v_5$ is a legitimate formula, as stupid as it may seem, $(= v_5 v_5)$ is not a legitimate formula as the only time you may introduce brackets to a formula is if you add a negation, $(\neg \alpha)$, or implication, $(\alpha \to \beta)$. The correct way to write this would be $(\neg \forall v_5 = v_5 v_5)$.

In general, if you:

- Don't see a $\neg$ immediately after the open bracket (, and;
- There is no $\to$ between the two brackets;

then it is not a legitimate formula.

(5) $\underline{< + - 1 \mid v_1 v_3}$: This is not a legitimate formula defined by the language of Example 5.1, $\mathcal{L}_{NT}$.

$$\begin{aligned}< +01 \mid v_1 v_3 &\to < +01(v_1 \mid v_3) && \text{Dissecting the } \mid \\ &\to < (0+1)(v_1 \mid v_3) && \text{Dissecting the } + \\ &\to (0+1) < (v_1 \mid v_3) && \text{Dissecting the } <\end{aligned}$$

This is not legitimate, as $v_1 \mid v_3$ is not a term. Refer to Definition 5.3(1).

(6) $\underline{(v_3 = v_3 \to \forall v_5\, v_3 = v_5)}$: This is another formula which humans can easily read, but formally it is written incorrectly. For instance, the formal way to write "$v_3 = v_5$" is $= v_3 v_5$.

(7) $\forall v_6(= v_6 0 \to \forall v_9(\neg \mid v_9 v_6))$: This is a legitimate formula defined by the language of Example 5.1, $\mathcal{L}_{NT}$.

$$\begin{aligned} & \forall v_6(= v_6 0 \to \forall v_9(\neg \mid v_9 v_6)) \\ \to\ & \forall v_6(= v_6 0 \to \forall v_9(\neg(v_9 \mid v_6))) && \text{Dissecting the } \mid \\ \to\ & \forall v_6(v_6 = 0 \to \forall v_9(\neg(v_9 \mid v_6))) && \text{Dissecting the } = \end{aligned}$$

If you look closely and try to interpret the formula, the formula is false: If $v_6 = 0$, then surely every integer/element divides 0. However, there's a difference between a formula being *legitimate*, and a formula being *false* (when interpreted).

(8) $\forall v_8 < +11 v_4$: This is a legitimate formula defined by the language of Example 5.1, $\mathcal{L}_{NT}$.

$$\begin{aligned} \forall v_8 < +11 v_4 &\to \forall v_8 < (1+1) v_4 && \text{Dissecting the } + \\ &\to \forall v_8((1+1) < v_4) && \text{Dissecting the } < \end{aligned}$$

This formula looks strange: The quantified v_8 doesn't appear inside the formula. However, we do not actually need the quantified variable to appear anywhere else - it simply becomes obsolete.

Activity 2.

Review Problem 5.8. Write down a formula of the given language expressing the given informal statement.

Solution

It's best to approach these problems by writing out the "dissected" version of the formulas first, then rewrite them in the formal way.

(1) "Addition is associative" in $\mathcal{L}_F$: The dissected version is:

$$\forall x \, \forall y \, \forall z \, ((x+y)+z = x+(y+z))$$

Thus, the formal version is:

$$\forall x \, \forall y \, \forall z \, (= + + xyz + x + yz)$$

(2) <u>"There is an empty set" in $\mathcal{L}_S$</u> The dissected version is:

$$\exists x \, \forall y \, (y \notin x)$$

Thus, the formal version is:

$$\exists x \, \forall y \, (\neg \in yx)$$

(3) <u>"Between any two distinct elements there is a third element" in $\mathcal{L}_O$</u>: The dissected version is:

$$\forall x \, \forall y \, (x \neq y \to \exists z \, ((x < z \wedge z < y) \vee (y < z \wedge z < x)))$$

Thus, the "formal" version is:

$$\forall x \, \forall y \, (\neg = xy \to \exists z \, ((< xz \wedge < zy) \vee (< yz \wedge < zx)))$$

The reason why I put "formal" in quotes is because $\wedge$ and $\vee$ are not part of the formal language, but are abbreviations. See "Common Conventions" in page 30 of the book. To write it in a fully formal way, you can replace them with $(\neg(\alpha \to (\neg\beta)))$ and $((\neg\alpha) \to \beta)$ respectively.

(4) <u>"$n^0 = 1$ for all n different from 0" in $\mathcal{L}_N$</u>: The dissected version is:

$$\forall n \, (n \neq 0 \to E(n, 0) = 1)$$

Thus, the formal version is:

$$\forall n \, (\neg = n0 \to = En01)$$

(5) <u>"There is only one thing" in $\mathcal{L}_=$</u>: The dissected version is:

$$\exists x \, \forall y \, (y = x)$$

Thus, the formal version is:

$$\exists x \, \forall y \, (= yx)$$

Activity 3.

Review slide 9, and interpret what each statements written in the language of set theory means.

Solution

This wasn't exactly stated in the notes, but in the language of set theory, elements are usually called *sets*. We shall prvide an interpretation for (1), (2) and (4) - (3) and (5) are left for you to ponder.

(1) $\exists x\, \forall y (\neg\, y \in x)$: This formula is saying that:

> There is a set x such that for every set y, y is not an element of x.

In other words, there exists a set x such that x has no elements. This is why it's called the Axiom of Empty Set.

(2) $\forall x\, \forall y\, \exists z\, \forall w (w \in z \leftrightarrow w \in x \wedge w \in y)$: This formula is saying that:

> For every sets x and y, there exists some set z such that for every set w, $w \in z$ if and only if $w \in x$ and $w \in y$.

The statement "$w \in z$ if and only if $w \in x$ and $w \in y$" is saying that z contains precisely the elements that are in both x and y, i.e. $z = x \cap y$. Therefore, this statement asserts that for every sets x and y, there exists a set z such that $z = x \cap y$. This is a special case of the Axiom of Intersection.

(4) $\forall x\, \exists y\, \forall z (z \in y \leftrightarrow \forall w (w \in x \rightarrow z \in w))$: This formula is saying that:

> For every set x, there exists some set y such that for every set z, $z \in y$ if and only if $z \in w$ for every $w \in x$.

Making it more "English" once again, we get:

> Given a set x, we can always find a set y, such that y contains precisely the sets z in which $z \in w$ for all $w \in x$.

Give some time to think and realise that the statement "the sets z in which $z \in w$ for all $w \in x$" is saying that "the sets z in $\bigcap_{w \in x} w$", or sometimes simply written as $\bigcap x$. Therefore, this statement is basically saying that for every set x, there exists a set y such that $y = \bigcap x$.