MAT337 Introduction to Real Analysis - Fall 2025
Week 12 Tutorial

Problem 8.1.F.

Show that $f_n(x) = n\sin(\frac{x}{n})$ converges uniformly on $[-R, R]$ for any finite R but does not converge uniformly on $\mathbb{R}$.

Solution

Let $g(x) = x$. We shall show that (f_n) converges uniformly to g on $[-R, R]$ for any $R > 0$, but not on $\mathbb{R}$.

Let $h_n(x) := f_n(x) - g(x) = n\sin(\frac{x}{n}) - x$. Note that h_n has the following properties:

(1) $h_n'(x) = \cos(\frac{x}{n}) - 1 \leq 0$ for all x, so h_n is a decreasing function.

(2) $h_n(-x) = -h_n(x)$. In particular, $h_n(0) = 0$.

Observe that $|h_n(x)| \leq |h_n(R)|$ for all $x \in [-R, R]$ – if $x \geq 0$, then $0 \geq h_n(x) \geq h_n(R)$, so $|h_n(x)| \leq |h_n(R)|$. If $x < 0$, then $|h_n(x)| = |h_n(-x)| \leq |h_n(R)|$. Therefore, for all $x \in [-R, R]$:

$$|h_n(x)| \leq |h_n(R)| = \left| n\sin\left(\frac{R}{n}\right) - R \right|.$$

Exercise. Show that $\lim_{n\to\infty} n\sin\left(\frac{R}{n}\right) = R$, by expressing the limit as a derivative.

Thus, for any $\varepsilon > 0$, we may let N be large enough so that for all $n \geq N$, $\left| n\sin\left(\frac{R}{n}\right) - R \right| < \varepsilon$. Then for all $n \geq N$:

$$\begin{aligned} \sup_{x\in[-R,R]} |f_n(x) - g(x)| &= \sup_{x\in[-R,R]} |h_n(x)| \\ &\leq |h_n(R)| \\ &< \varepsilon, \end{aligned}$$

so f_n converges to g uniformly on $[-R, R]$.

We now show that (f_n) does not converge uniformly on $\mathbb{R}$. Suppose for a contradiction that (f_n) converges uniformly on $\mathbb{R}$. Since (f_n) converges to g pointwise on

$\mathbb{R}$ (because every $x \in \mathbb{R}$ is in some closed interval $[-R, R]$), this would imply that (f_n) converges uniformly to g on $\mathbb{R}$. Thus, there exists some M such that for all $n \geq M$, $|f_n(x) - g(x)| < 1$ for all $x \in \mathbb{R}$ and $n \geq M$. But:

$$\left| f_n\left(\frac{n\pi}{n}\right) - g(n\pi) \right| = |0 - n\pi| = n\pi > 1,$$

which is a contradiction.

Remarks/Takeaways.

(1) If (f_n) converges to g uniformly, then (f_n) converges to g pointwise - that is, $\lim_{n\to\infty} f_n(x) = g(x)$ for all x. Therefore, one may compute the limit $\lim_{n\to\infty} f_n(x)$ to know what the uniform limit of (f_n) is.

(2) Here is a common method to show that a sequence of functions (f_n) does not converge uniformly on some set $S \subseteq \mathbb{R}$.

   (a) Check if (f_n) converges pointwise on S. If not, find a point $x \in S$ such that $\lim_{n\to\infty} f_n(x)$ doesn't exist.

   (b) If (f_n) converges pointwise on S, then define the function $g(x) := \lim_{n\to\infty} f_n(x)$ on S. Show that there exists some $\varepsilon > 0$ such that for all N there is some $n \geq N$ and $x \in S$ such that $|f_n(x) - g(x)| \geq \varepsilon$.

Problem 8.1.H.

Suppose that $f_n : [0,1] \to \mathbb{R}$ is a sequence of C^1 functions (i.e., functions with continuous derivatives) that converges pointwise to a function f. If there is a constant M such that $\|f_n'\|_\infty \leq M$ for all n, then prove that (f_n) converges to f uniformly.

Solution

By the completeness theorem for $C([0,1], \mathbb{R})$ (as $[0,1]$ is compact), it suffices to show that for all $\varepsilon > 0$, there exists some N such that for all $m, n \geq N$, $\|f_n - f_m\|_\infty < \varepsilon$.

Recall that if $\|f_n'\|_\infty \leq M$, then f_n is Lipschitz continuous with Lipschitz constant M. In other words, for all $x, y \in [0,1]$:

$$|f_n(x) - f_n(y)| \leq M|x - y|.$$

We fix some $\varepsilon > 0$. Let $K \geq \frac{3M}{\varepsilon}$ be any integer, and let $x_i = \frac{i}{K}$ for all $i \leq K$. We are defining x_i this way so that for every $x \in [0,1]$, there is some i such that $|x - x_i| < \frac{1}{K}$.

Since $(f_n(x_i))$ is Cauchy for all i, there exists some N large enough so that for all $m, n \geq N$ and $i \leq K$, $|f_n(x_i) - f_m(x_i)| < \frac{\varepsilon}{3}$. Then for any $x \in [0,1]$:

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f_m(x_i)| + |f_m(x_i) - f_m(x)| \\
&\leq M|x - x_i| + |f_n(x_i) - f_m(x_i)| + M|x - x_i| \\
&< M \cdot \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} + M \cdot \frac{\varepsilon}{3M} \\
&= \varepsilon.
\end{aligned}$$

Remarks/Takeaways.

(1) The idea of the proof can be summarised into the following steps:

  (a) Given an ε, we can fix finitely many points $x_0, \dots, x_K$ such that for any $x \in [0,1]$, x is very close to x_i for some i.

  (b) Since all the f_n's are Lipschitz continuous with the same Lipschitz constant, this implies that for every n, if x is very close to x_i then $f_n(x)$ is also very close to $f(x_i)$.

  (c) We apply the Cauchy property of $(f_n(x_i))$ for $i = 0, \dots, K$. Then for all $n, m \geq N$ and $x \in [0,1]$:

    i. $f_n(x)$ is very close to $f_n(x_i)$ (by (b) above).

    ii. $f_n(x_i)$ is very close to $f_m(x_i)$ (by the Cauchy property of $(f_n(x_i))$).

    iii. $f_m(x_i)$ is very close to $f_m(x)$ (by (b) above).

  Therefore, $f_n(x)$ is very close to $f_m(x)$.

(2) The compactness of the domain $[0,1]$ is necessary. Here is a counterexample with the domain $[0,1]$ replaced by $\mathbb{R}$ – let $f_n(x) = \frac{1}{n}x$. It's clear that $\lim_{n\to\infty} f_n(x) = 0$ for all x, so (f_n) converges to f pointwise. Furthermore, $\|f_n'\|_\infty = \frac{1}{n} \leq 1$ for all n.

  Exercise. Show that (f_n) does not converge to 0 uniformly on $\mathbb{R}$.

(3) It is possible to give a proof of Problem 8.1.H without using the completeness theorem for $C([0,1],\mathbb{R})$, but you need to first show that f is uniformly continuous. Once that is done, the proof is very similar, with f_m replaced by f (and some additional requirements on the choice of N).

Problem 8.2.F.

Let $f_n(x) = \frac{\arctan(nx)}{\sqrt{n}}$.

(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$, and show that (f_n) converges uniformly to f on $\mathbb{R}$.

(b) Compute $\lim_{n\to\infty} f_n'(x)$, and compare this with $f'(x)$.

(c) Where is the convergence of f_n' uniform? Prove your answer.

Solution

(a) We shall show that (f_n) converges to 0 uniformly on $\mathbb{R}$. Fix some $\varepsilon > 0$. Observe that $|\arctan(nx)| \leq \frac{\pi}{2}$ for all x. Thus, we may choose N large enough so that $\frac{\pi}{2N} < \varepsilon$. Then for all $n \geq N$:

$$\sup_{x\in\mathbb{R}} \left|\frac{\arctan(nx)}{\sqrt{n}}\right| \leq \frac{\pi}{2\sqrt{n}} < \varepsilon.$$

Thus, (f_n) converges to 0 uniformly on $\mathbb{R}$.

(b) For each n, we have:

$$f_n'(x) = \frac{1}{\sqrt{n}}\left(\frac{n}{1+n^2x^2}\right) = \frac{\sqrt{n}}{1+n^2x^2}.$$

Note that $(f_n'(x))$ converges to 0 for all $x \neq 0$ - for each x, we may apply the squeeze theorem to show that:

$$0 \leq \lim_{n\to\infty} \frac{\sqrt{n}}{1+n^2x^2} \leq \lim_{n\to\infty} \frac{\sqrt{n}}{n^2x^2} = \lim_{n\to\infty} \frac{1}{n^{\frac{3}{2}}x^2} = 0.$$

However, $f_n'(0) = \sqrt{n}$, so $(f_n'(0))$ diverges to $+\infty$.

(c) We shall show that (f_n') -doesn't converge to 0 uniformly on $\mathbb{R} \setminus \{0\}$, but it converges to 0 uniformly $\mathbb{R} \setminus (-r, r)$ for all $r > 0$.

Suppose that (f_n') converges to 0 uniformly on $\mathbb{R} \setminus \{0\}$, so there exists some N such that for all $n \geq N$, $|f_n'| < \frac{1}{2}$. If $x = \frac{1}{n} > 0$, then:

$$|f_n'(x)| = \frac{\sqrt{n}}{1+n^2(1/n)^2} = \frac{\sqrt{n}}{2} \geq \frac{1}{2},$$

which is a contradiction.

Fix some $r > 0$. We now show that (f_n') converges to 0 uniformly on $\mathbb{R} \setminus (-r, r)$. Observe that f_n' is decreasing for all n, so:

$$|f_n'(x)| = \frac{\sqrt{n}}{1 + n^2 x^2} \leq \frac{1}{n^{\frac{3}{2}} x^2} \leq \frac{1}{n^{\frac{3}{2}} r^2}.$$

Fix $\varepsilon > 0$. Let N be large enough so that $\frac{1}{n^{\frac{3}{2}} r^2} < \varepsilon$. Then $|f_n'(x)| < \varepsilon$ for all x, so (f_n') converges to 0 uniformly on $\mathbb{R} \setminus (-r, r)$.

Remarks/Takeaways.

(1) Notice that the method we used to show that (f_n') doesn't converge uniformly on $\mathbb{R} \setminus \{0\}$ (i.e. proof by contradiction) is the same as that of Problem 8.1.F.