MAT337 Introduction to Real Analysis - Fall 2025
Week 6 Tutorial

This tutorial focuses on closed and open subsets of $\mathbb{R}^n$, and the compactness property. Feel free to ask me any questions about this document in person.

Definition. Let $A \subseteq \mathbb{R}^n$.

(1) A point $\vec{x}$ is a **limit point** of A is there exists a sequence $(\vec{a}_k)_{k=1}^{\infty}$, with $\vec{a}_k \in A$ for all k, such that $\vec{x} = \lim_{k\to\infty} \vec{a}_k$.

(2) A is **closed** if it contains all of its limit points.

Problem 4.3.A.

Find the closure of the following sets:

(a) $\mathbb{Q}$.

(b) $\{(x,y) \in \mathbb{R}^2 : xy < 1\}$.

(c) $\{(x, \sin(\frac{1}{x})) : x > 0\}$.

(d) $\{(x,y) \in \mathbb{Q}^2 : x^2 + y^2 < 1\}$.

Solution

(a) The closure is $\mathbb{R}$, as every real number is a limit of a sequence of rational numbers (see Problem 2.5.G).

(b) The closure is $\{(x,y) \in \mathbb{R}^2 : xy \leq 1\}$.

Suppose that $(x_k, y_k)_{k=1}^{\infty}$ is a sequence in the set, and $\lim_{k\to\infty}(x_k, y_k) = (x, y)$. We need to show that $xy \leq 1$. It suffices to show that for all $\varepsilon \in (0,1)$ (i.e. $0 < \varepsilon < 1$), $xy < 1 + \varepsilon$. Let k be large enough so that $\|(x_k, y_k) - (x, y)\| <$

$\frac{\varepsilon}{|x|+|y|+1}$. Then:

$$\begin{aligned}
xy &= (xy - x_k y_k) + x_k y_k \\
&< |xy - x_k y_k| + 1 \\
&\leq |xy - x_k y| + |x_k y - x_k y_k| + 1 \\
&= |y||x - x_k| + |x_k||y - y_k| + 1 \\
&\leq |y||x - x_k| + (|x_k - x| + |x|)|y - y_k| + 1 \\
&< |y|\frac{\varepsilon}{|x| + |y| + 1} + (\varepsilon + |x|)\frac{\varepsilon}{|x| + |y| + 1} + 1 \\
&= \frac{|x| + |y| + \varepsilon}{|x| + |y| + 1}\varepsilon + 1 \\
&< \varepsilon + 1.
\end{aligned}$$

Conversely, suppose that $(x, y) \in \mathbb{R}^2$ is such that $xy \leq 1$. We need to find some sequence $(x_k, y_k)_{k=1}^{\infty}$ such that $x_k y_k < 1$ for all k, and $\lim_{k\to\infty}(x_k, y_k) = (x, y)$. If $xy < 1$, then we may let $(x_k, y_k) = (x, y)$ for all k. Otherwise, $xy = 1$, so in particular we have that $x \neq 0$. We consider two cases.

(i) If $x > 0$ (so $y > 0$ as well), then for each k we let $x_k = x - \frac{1}{k}$ and $y_k = y$. It's not hard to see that $\lim_{k\to\infty}(x_k, y_k) = (x, y)$, and for each k, $x_k y_k < xy = 1$.

(ii) **Exercise.** Find the sequence in the case where $x < 0$ (so $y < 0$ as well).

(c) The closure is $\{(x, \sin(\frac{1}{x})) : x > 0\} \cup \{(0, y) : -1 \leq y \leq 1\}$.

Suppose that $(x_k, y_k)_{k=1}^{\infty}$ is a sequence in the set, and $\lim_{k\to\infty}(x_k, y_k) = (x, y)$. We need to show that either $x > 0$ and $y = \sin(\frac{1}{x})$, or $x = 0$ and $-1 \leq y \leq 1$.

(i) Suppose that $x > 0$. Let $\varepsilon > 0$. Since $\lim_{k\to\infty} x_k = x$, we have that $\lim_{k\to\infty} \frac{1}{x_k} = \frac{1}{x}$, so there exists some k such that $|\frac{1}{x_k} - \frac{1}{x}| < \varepsilon$. We shall use the trigonometric identity:

$$\sin(A) - \sin(B) = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right),$$

and the inequality $\sin(x) \le x$ for all $x \ge 0$. then:

$$\begin{aligned}
\left|\sin\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x_k}\right)\right| &= \left|2\sin\left(\frac{1}{2}\left(\frac{1}{x} - \frac{1}{x_k}\right)\right)\cos\left(\frac{1}{2}\left(\frac{1}{x} + \frac{1}{x_k}\right)\right)\right| \\
&= 2 \cdot \left|\sin\left(\frac{1}{2}\left(\frac{1}{x} - \frac{1}{x_k}\right)\right)\right| \cdot \left|\cos\left(\frac{1}{2}\left(\frac{1}{x} + \frac{1}{x_k}\right)\right)\right| \\
&\le 2 \cdot \left|\frac{1}{2}\left(\frac{1}{x} - \frac{1}{x_k}\right)\right| \cdot 1 \\
&= \left|\frac{1}{x_k} - \frac{1}{x}\right| \\
&< \varepsilon
\end{aligned}$$

(ii) If $x = 0$, then since $|y_k| = |\sin(\frac{1}{x_k})| \le 1$ for all k, by the squeeze theorem we have that $|y| \le 1$ as well.

Now suppose that $(0, y) \in \mathbb{R}^2$ and $-1 \le y \le 1$. We need to find some sequence $(x_k, y_j)_{k=1}^{\infty}$ such that $y_k = \sin(\frac{1}{x_k})$, and $\lim_{k\to\infty}(x_k, y_k) = (0, y)$. Since $-1 \le y \le 1$, there exists some $0 < d \le 2\pi$ such that $y = \sin(d)$. Let $x_k = \frac{1}{2k\pi + d}$. Then $\lim_{k\to\infty} x_k = 0$ as $\lim_{k\to\infty}(2k\pi + d) = +\infty$, and for all k:

$$y_k = \sin\left(\frac{1}{\frac{1}{2k\pi+d}}\right) = \sin(2k\pi + d) = \sin(d) = y,$$

so $(y_k)_{k=1}^{\infty}$ is the constant sequence where every term is y.

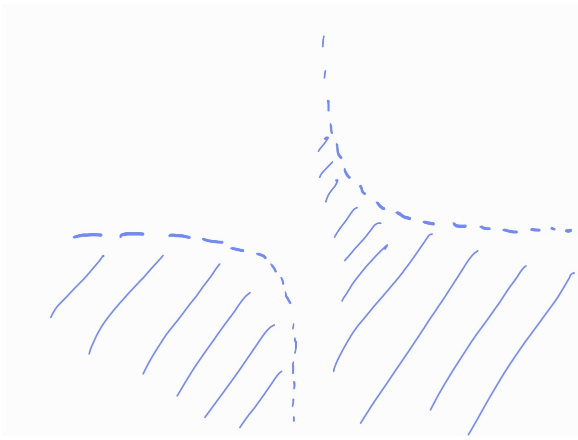
(d) The closure is $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$.

Exercise. Solve Problem 4.3.A(d).

Remarks/Takeaways.

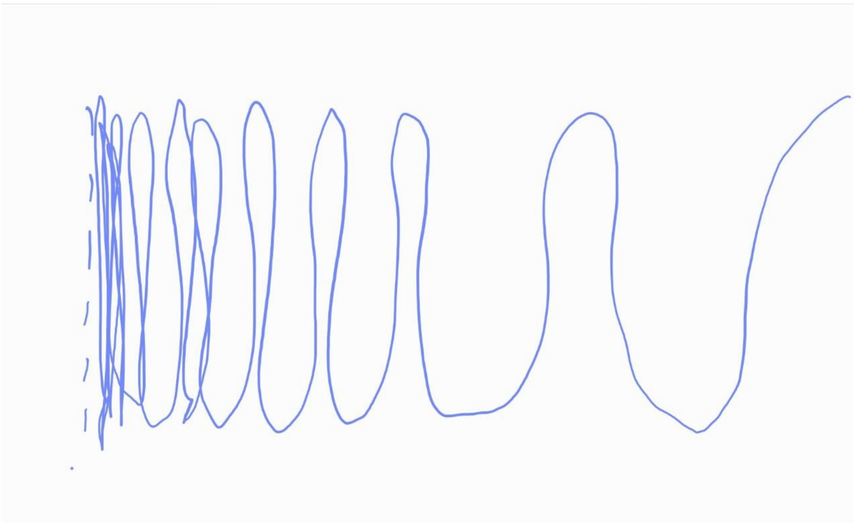
(1) It's crucial for one to develop the skill to quickly see what the closure of a set is. Finding the closure of a set amounts to finding the "boundary" of a set.

(b) For Q4.3.A(b), the set looks like this:



The dotted line (i.e. the set $\{(x, y) \in \mathbb{R}^2 : xy = 1\}$) is not part of the set, but it's a "boundary" of it. Therefore, the dotted line is included in the closure.

(c) For Q4.3.A(c), the set looks like this:



Again, dotted line (i.e. the set $\{(0, y) : -1 \leq y \leq 1\}$) is not part of the set, but the points in the set "gets closer and closer" to the dotted line, so it's included in the closure.

(2) If a set has the strict inequality $<$, the closure likely has the equality replaced with $\leq$ (see (b), (c) and (d)).

Problem 4.3.B.

Let $(\vec{a}_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$ with $\lim_{k\to\infty} \vec{a}_k = \vec{a}$. Show that $\{\vec{a}_k : k \geq 1\} \cup \{\vec{a}\}$ is closed.

Solution

Let $A := \{\vec{a}_k : k \geq 1\} \cup \{\vec{a}\}$, and let $\vec{x}$ be a limit point of A. We need to show that $\vec{x} \in A$. Let $(\vec{b}_l)_{l=1}^{\infty}$ be a sequence, with $\vec{b}_l \in A$ for all l, such that $\lim_{l\to\infty} \vec{b}_l = \vec{x}$. We consider two cases.

(1a) Suppose that there are infinitely many l such that $\vec{b}_l = \vec{a}$. Let $(\vec{b}_{l_m})_{m=1}^{\infty}$ be a subsequence (i.e. $l_1 < l_2 < \cdots$) such that $\vec{b}_{l_m} = \vec{a}$ for all m. Then $\lim_{m\to\infty} \vec{b}_{l_m} = \vec{a}$. But since $\lim_{l\to\infty} \vec{b}_l = \vec{x}$, every subsequence of $(\vec{b}_l)_{l=1}^{\infty}$ also converges to $\vec{x}$, so $\vec{x} = \vec{a} \in A$.

(1b) Suppose that there is some k such that there are infinitely many l in which $\vec{b}_l = \vec{a}_k$. By a similar argument to Case (1a), we have that $\vec{x} = \vec{a}_k \in A$.

(2) Suppose that neither (1a) nor (1b) are true. We define a subsequence $(\vec{b}_{l_m})_{m=1}^{\infty}$ inductively as follows: Let $\vec{b}_{l_0}$ be any term such that $\vec{b}_{l_0} = \vec{a}_{k_0}$ for some k_0. Suppose that $\vec{b}_{l_m}$ has been defined and $\vec{b}_{l_m} = \vec{a}_{k_m}$ for some k_m.

Exercise. Show that there exists some $l_{m+1} > l_m$ such that $\vec{b}_{l_{m+1}} = \vec{a}_{k_{m+1}}$ for some $k_{m+1} > k_m$.

Then $(\vec{b}_{l_m})_{m=1}^{\infty} = (\vec{a}_{k_m})_{m=1}^{\infty}$ is a subsequence of $(\vec{a}_k)_{k=1}^{\infty}$, and since $\lim_{k\to\infty} \vec{a}_k = \vec{a}$, we have that $\lim_{m\to\infty} \vec{a}_{k_m} = \vec{a}$ as well. Therefore, $\vec{x} = \vec{a} \in A$.

Remarks/Takeaways.

(1) If the question asks you to show that a set is closed, proving it by definition is a common approach.

(2) Do not conclude hastily that $(\vec{b}_l)_{l=1}^{\infty}$ is a subsequence of $(\vec{a}_k)_{k=1}^{\infty}$, as it may not be.

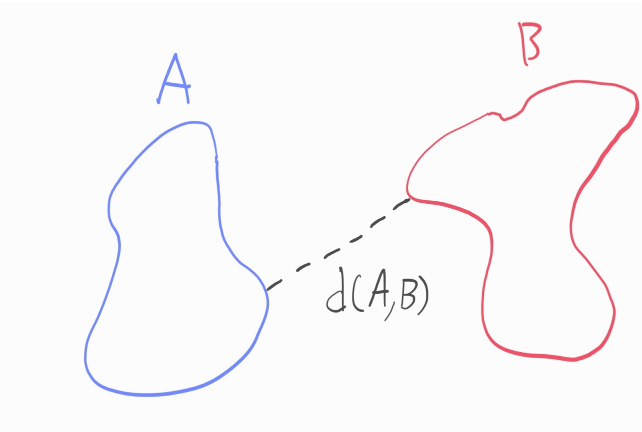
Problem 4.4.I.

Let A and B be *disjoint* closed subsets of $\mathbb{R}^n$. Define:

$$d(A, B) := \inf\{\|\vec{a} - \vec{b}\| : \vec{a} \in A, \vec{b} \in B\}.$$

Remark.

(1) $d(A, B)$ may be viewed as the shortest distance needed to travel between two closed sets (but not always! See 4.4.I(c)). Here's a picture:



4.4.I(a).

If $\vec{a}$ is a singleton, show that $d(A, B) > 0$.

Solution

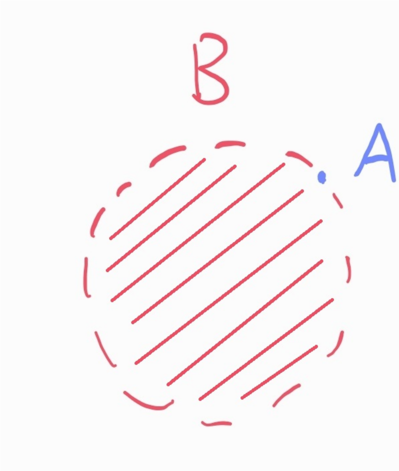
We shall show that if $d(\{\vec{a}\}, B) = 0$, then $\vec{a} \in B$, contradicting that A and B are disjoint. Since B is closed, it suffices to show that $\vec{a}$ is a limit point of B. That is, there exists a sequence $(\vec{a}_k)_{k=1}^{\infty}$ in B such that $\lim_{k\to\infty} \vec{a}_k = \vec{a}$.

Since $d(\{\vec{a}\}, B) = 0$, for all $\varepsilon > 0$ there exists some $\vec{b} \in B$ such that $\|\vec{a} - \vec{b}\| < \varepsilon$. In particular, for any k there exists some $\vec{a}_k \in B$ such that $\|\vec{a}_k - \vec{a}\| < \frac{1}{k}$. Then $0 \leq \|\vec{a}_k - \vec{a}\| < \frac{1}{k}$ for all k, so by the squeeze theorem, we have that $\lim_{k\to\infty} \|\vec{a}_k - \vec{a}\| = 0$, i.e. $\lim_{k\to\infty} \vec{a}_k = \vec{a}$.

Remarks/Takeaways.

(1) This is not the first time you see us taking a sequence (or subsequence) in which each $\|\vec{a}_k - \vec{a}\|$ is bounded by some other sequence which converges to zero. This is a useful technique which will appear from time to time.

(2) The statement is false if we do not assume that B is closed - that is, it is possible that $d(\vec{a}, B) = 0$ and $\vec{a} \notin B$ if B is not closed. Here is an example:



Equations:

(a) $A = \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$.

(b) $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.

4.4.I(b).

If A is compact, show that $d(A, B) > 0$.

Definition. Let $A \subseteq \mathbb{R}^n$. A is **compact** if for every sequence $(\vec{a}_k)_{k=1}^{\infty}$ of points in A, there exists a convergent subsequence $(\vec{a}_{k_i})_{i=1}^{\infty}$ with $\lim_{i\to\infty} \vec{a}_{k_i} = \vec{a} \in A$.

Solution

Exercise. Show that $d(A, B) = \inf\{d(\{\vec{a}\}, B) : \vec{a} \in A\}$. If you need a hint, see the Remarks/Takeaways after the solution.

We shall show that if $d(A, B) = 0$, then A is not compact. Since $\inf\{d(\{\vec{a}\}, B) : \vec{a} \in A\} = 0$, for any k there exists some $\vec{a}_k \in B$ such that $d(\{\vec{a}_k\}, B) < \frac{1}{k}$. Then $(\vec{a}_k)_{k=1}^{\infty}$, and since A is compact, there exists a convergent subsequence $(\vec{a}_{k_i})_{i=1}^{\infty}$ with $\lim_{i\to\infty} \vec{a}_{k_i} = \vec{a} \in A$. We shall show that $d(\vec{a}, B) = 0$, contradicting Problem 4.4.I(a).

It suffices to show that $d(\vec{a}, B) < \varepsilon$ for all $\varepsilon > 0$. Fix any $\varepsilon > 0$. Since $\lim_{i\to\infty} \vec{a}_{k_i} = \vec{a}$, there exists some k_i large enough such that $\frac{1}{k_i} < \frac{\varepsilon}{2}$, and $\|\vec{a} - \vec{a}_{k_i}\| < \frac{\varepsilon}{2}$. Then for all $\vec{b} \in B$:

$$\begin{aligned}\|\vec{a} - \vec{b}\| &\leq \|\vec{a} - \vec{a}_{k_i}\| + \|\vec{a}_{k_i} - \vec{b}\| \\ &< \frac{\varepsilon}{2} + \|\vec{a}_{k_i} - \vec{b}\|.\end{aligned}$$

Taking infimum across $\vec{b} \in B$, we have that:

$$\begin{aligned}d(\{\vec{a}\}, B) &= \inf\{\|\vec{a} - \vec{b}\| : \vec{b} \in B\} \\ &< \frac{\varepsilon}{2} + \inf\{\|\vec{a}_{k_i} - \vec{b}\| : \vec{b} \in B\} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Therefore, $d(\vec{a}, B) < \varepsilon$. Since ε is arbitrary, we have that $d(\vec{a}, B) = 0$.

Remarks/Takeaways.

(1) The exercise at the beginning of the solution is an excellent practice in proving equalities by showing one is bounded by the other, and vice versa.

  (a) Show that for all $\vec{a} \in A$, $d(A, B) \leq d(\{\vec{a}\}, B)$. By taking infimum across $\vec{a} \in A$, you may conclude that $d(A, B) \leq \inf\{d(\{\vec{a}\}, B) : \vec{a} \in A\}$.

  (b) Show that for all $\varepsilon > 0$, there exists some $\vec{a} \in A$ such that $d(\{\vec{a}\}, B) \leq d(A, B) + \varepsilon$. Explain why this implies that $\inf\{d(\{\vec{a}\}, B) : \vec{a} \in A\} \leq d(A, B)$.

4.4.I(c).

Find an example of two disjoint closed sets in $\mathbb{R}^2$ with $d(A, B) = 0$.

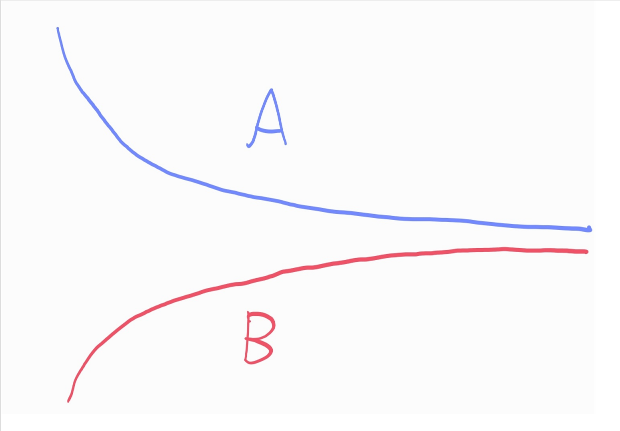
Solution

Consider the following two sets:

(1) $A = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y = \frac{1}{x}\}$.

(2) $B = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y = -\frac{1}{x}\}$.

In other words, A and B are the graphs of the functions $y = \frac{1}{x}$ and $y = -\frac{1}{x}$ respectively. Here's a picture:



We need to show that A and B are closed, and $d(A, B) = 0$.

We first show that A is closed. Let $(a, b) \in \mathbb{R}^2$ be a limit point of A, i.e. there exists some sequence $((x_k, y_k))_{k=1}^{\infty}$ in A such that $\lim_{k\to\infty}(x_k, y_k) = (a, b)$. This implies that $\lim_{k\to\infty} x_k = a$ and $\lim_{k\to\infty} y_k = b$. Since $y_k = \frac{1}{x_k}$, we have that $b = \lim_{k\to\infty} \frac{1}{x_k} = \frac{1}{a}$, so $(a, b) = (a, \frac{1}{a}) \in A$.

Exercise. Using the idea in the above paragraph, convince yourself that B is also closed.

We now show that $d(A, B) = 0$. It suffices to show that for all $\varepsilon > 0$, $d(A, B) < \varepsilon$. For each $\varepsilon > 0$, let $x > \frac{2}{\varepsilon}$ be any number large enough. Then $(x, \frac{1}{x}) \in A$ and $(x, -\frac{1}{x}) \in B$, and:

$$\begin{aligned}
\left\|\left(x, \frac{1}{x}\right) - \left(x, -\frac{1}{x}\right)\right\| &= \left\|\left(0, \frac{2}{x}\right)\right\| \\
&= \frac{2}{x} \\
&< \varepsilon.
\end{aligned}$$

Therefore, $d(A, B) \leq \left\|\left(x, \frac{1}{x}\right) - \left(x, -\frac{1}{x}\right)\right\| < \varepsilon$.

Remarks/Takeaways.

(1) Recall by the Heine-Borel theorem that a set is compact iff it is closed and bounded. A and B are examples of sets which are closed but unbounded.

(2) If you try to repeat the proof of Problem 4.4.I(b) with A and B, the sequence $(\vec{a}_k)_{k=1}^{\infty}$ does not have a convergent subsequence as the x-coordinate of $\vec{a}_k$ tends to

infinity. If the set is compact, the boundedness property prevents the coordinates from "escaping" to infinity, thus guaranteeing us a convergent subsequence.