MAT337 Introduction to Real Analysis - Fall 2025
Week 8 Tutorial (Addendum)

There was a small issue with the solution to Problem 5.1.D I gave in the tutorial. I used the fact that for all $t > -1$:

$$\frac{t}{1+t} \leq \log(1+t) \leq t.$$

Letting $t = xy$, we have that at a neighbourhood around $(0, y_0)$:

$$\frac{xy}{1+xy} \leq \log(1+xy) \leq xy.$$

Then I claimed that this implies:

$$\frac{y}{1+xy} \leq \frac{\log(1+xy)}{x} \leq y.$$

This is only true when $x > 0$, so we need to address the case when $x < 0$. The correct argument can be found below. I have also included a proof of the inequality involving the logarithm for completeness.

Problem 5.1.D.

Prove that f is continuous at $(0, y_0)$, where f is defined on $\mathbb{R}^2$ by:

$$f(x,y) = \begin{cases} (1+xy)^{\frac{1}{x}}, & \text{if } x \neq 0, \\ e^y, & \text{if } x = 0. \end{cases}$$

Solution

The key observation is that:

$$\lim_{(x,y)\to(0,y_0)} (1+xy)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{\log(1+xy)}{x}} = e^{\lim_{(x,y)\to(0,y_0)} \frac{\log(1+xy)}{x}}.$$

So it suffices to show that:

$$\lim_{(x,y)\to(0,y_0)} \frac{\log(1+xy)}{x} = y_0.$$

Exercise. Show that for all $t > -1$:

$$\frac{t}{1+t} \leq \log(1+t) \leq t.$$

Proof. We first show that $\log(1+t) \leq t$ for all $t > -1$. Let $g(t) = \log(1+t)$ and $f(t) = t$. Then $g'(t) = \frac{1}{1+t}$ and $f'(t) = 1$. Note that $g(0) = 0 = f(0)$.

(1) Since $g'(t) < f'(t)$ for all $t > 0$ and $g(0) = f(0)$, we have that $g(t) < f(t)$ for all $t > 0$.

(2) Since $g'(t) > f'(t)$ for all $-1 < t < 0$ and $g(0) = f(0)$, we have that $g(t) < f(t)$ for all $t > 0$.

Note that if $t > -1$, then $-\frac{t}{1+t} > -1$. Therefore, applying the same inequality with t substituted as $-\frac{t}{1+t}$, we get:

$$\begin{aligned}
\log\left(1 - \frac{t}{1+t}\right) \leq -\frac{t}{1+t} &\implies \log\left(\frac{1}{1+t}\right) \leq -\frac{t}{1+t} \\
&\implies -\log(1+t) \leq -\frac{t}{1+t} \\
&\implies \frac{t}{1+t} \leq \log(1+t).
\end{aligned}$$

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Letting $t = xy$, we have that at a neighbourhood around $(0, y_0)$:

$$\frac{xy}{1+xy} \leq \log(1+xy) \leq xy.$$

Therefore:

(1) If $x > 0$, then:

$$\frac{y}{1+xy} \leq \frac{\log(1+xy)}{x} \leq y.$$

(2) If $x < 0$, then:

$$y \leq \frac{\log(1+xy)}{x} \leq \frac{y}{1+xy}.$$

This means that, for any x:

$$\min\left\{y, \frac{y}{1+xy}\right\} \leq \frac{\log(1+xy)}{x} \leq \max\left\{y, \frac{y}{1+xy}\right\}.$$

Observe that $\lim_{(x,y)\to(0,y_0)} y = y_0$ and $\lim_{(x,y)\to(0,y_0)} \frac{y}{1+xy} = y_0$. Since max and min are continuous functions (see Exercise 5.3.J, it's a homework problem!):

$$\lim_{(x,y)\to(0,y_0)} \min\left\{y, \frac{y}{1+xy}\right\} = y_0.$$
$$\lim_{(x,y)\to(0,y_0)} \max\left\{y, \frac{y}{1+xy}\right\} = y_0.$$

By the squeeze theorem, $\lim_{(x,y)\to(0,y_0)} \frac{\log(1+xy)}{x} = y_0$.